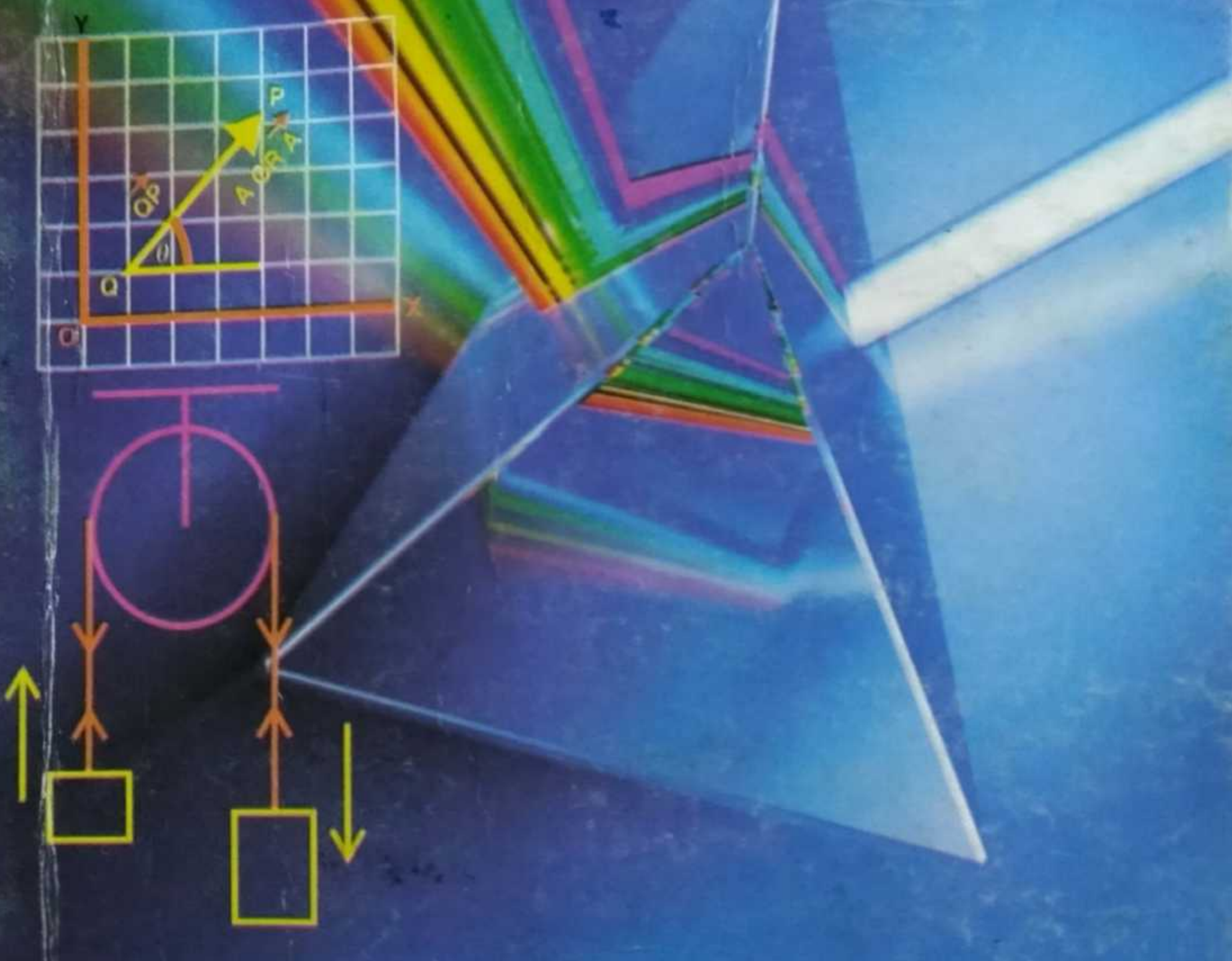
FUNDAMENTAL OF

PHYSICS

FOR CLASS-XI



SINDH TEXTBOOK BOARD,
JAMSHORO, SINDH

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OF

PHYSICS

FOR

CLASS XI

SINDH TEXTBOOK BOARD, JAMSHORO.

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CONTENTS

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

CHAPTER 1

The Scope of Physics

1.1 DEFINITION OF PHYSICS

The colours in the rainbow, the dropping of a mango from the branch of a tree, the rusting of an iron piece, the growing of Plants, the motion of bodies, the formation of the solar system, etc. are all phenomena of nature. Such Phenomena lead us to the study of science.

The subject of science is classified into two main branches :

(i) the physical sciences and

(ii) the biological sciences.

The biological science deal with living things where as the physical sciences are concerned with the properties and behaviour of non-living matter. The branch of Physical science which deals with the interaction of matter and energy is called physics. This is based on experimental observations and quantitative measurements.

Physics is therefore an experimental science which depends heavily upon the objective observations and measurement of natural phenomena.

The history of physics is as old as the history of mankind. Even the cave man was aware of the production of fire by rubbing two stones together. The Chinese for the first time manufactured paper (Papytue). Egyptian used to measure the flood level in the river Nile.

The people of Euphrates and Tigris valleys were aware of calendar and they had the knowledge of geometry. The people of Indus valley were the pioneers of decimal system.

The history of Greeks is full of inventions and discoveries in the field of all sciences, and specially in physics. Archimedes principle is still an important topic in elementary books of physics. He invented lever and screw. Pythagoras, Galen, Ptolemy, and others are famous in the field of mathematics, astronomy, medicine,etc.

The contribution of Muslims in the field of science in general and in physics in particular will be described in detail in section 1.3.

Significant contribution was made by Galileo-Galilei (1564-1642) through his work on the laws of motion in the presence of constant acceleration. Johanne Kepler (1571-1630) was his contemporary and presented Kepler's law of planetary motion.

Prior to 1900 AD, physics comprised of mechanics, sound, light, heat, magnetism and electricity. The new era of modern physics began near the end of 19th century. There are two main branches of physics now, namely Classical Physics and Quantum Physics. Einstein theory of relativity not only revolutionized the traditional concept of mass, time and energy but also modified Newton's laws of motion for describing the bodies moving with the speed comparable with the speed of light. The other branches of Physics are as follows:

(i) Particle Physics
(ii) Nuclear Physics
(iii) Molecular and Atomic Physics
(iv) Plasma Physics
(v) Astro Physics
(vi) Medical Physics.
(vii) Solid State Physics.

1.2. ISLAM AND SCIENCE

In the field of scientific research the strong incentive comes from no other book and no other philosopher as it comes from the Holy Quran. We are told in Surah Nooh.

"Do you not see how God made seven heavens One above the other ?"

"And He has placed the moon as a light in them.

He has made the sun as a lamp

Surah Nooh Ayah 15 and 16 (reference)

We are further reminded in Surah Al-Imran.

Surely in the making of the heavens & the earth,
And the alternations of the night and the day.

There are signs for people of understanding-

Those who remember God standing and sitting
and on their sides;

And those who use thought about the make of
the heavens and the earth.

" Our Lord! Thou hast not made this in vain,

Glory to be Thee,

So save us from the agony of the fire

Surah Al-Imran Ayah 190 and 191

It is again emphasized

God is He who has ordered the ocean for you that the ships may sail there, in together with his command.

And you may seek of his grace and in order that you may give thanks

And He has ordered for your benefit whatever is in the heavens and whatever is in the earth, all is from Him.

Most surely there are signs in this for a people who reflect

Surah Al-Jasiya (Ayah 12 and 13)

1.3 CONTRIBUTION TO PHYSICAL SCIENCES BY THE ISLAMIC WORLD.

Inspired by the Quranic verses and teachings of the Holy Prophet Muhammad (S.A.S), the muslims translated the ancient book and initiated the original search at the end of ninth century A.D. The famous chemist, Jabir-bin-Hayyan, for the first time, derived various laws in chemistry on the basis of experiments. Al-Battani made calculations in connection with solar system, change in seasons, eclipses of moon and sun and other astronomical phenomena. Remarkable and distinguished contributions were made

by most outstanding Muslim scientist, named, Al-Khawarizmi. He was founder of Analytical Algebra. His famous treatise; "Hisabul-Jubr-wal-Muqabla", is the first work on this subject. He invented the term Logarithm (algorithm). Omer Khayyam was one of the prominent mathematicians of the Islamic world. Bertrand Russel remarked about him with the following:

"Omer Khayyam is the only man known to me who was both a poet and mathematician".

Ibn-al-Haitham was a great physicist of the Islamic world. He wrote many books. His master piece work was the book named "Kitabul-Manazir". It is the first comprehensive book on light. He developed the laws of reflection and refraction. He constructed the pin hole camera. The most wonderful chapter of his book is on human eye. The information given in it is still valid and correct.

Al-Razi was the most prominent and greatest physician. He wrote about 200 original monographs, half of which pertained to medicine.

Abu-Rehan Al-Beruni was contemporary of Ibn-e-Sina. He was an astronomer, physician and mathematician. He also lived in India for sometime during the reign of Moghul emperor AKBAR. His famous book Kitab-ul-Qanoon-ul-Masoodi is considered as Encyclopaedia of Astronomy.

Yakoob Bin Ishaq Al-Kindi wrote many books on mathematics, astronomy, medicine and other subjects:

Ibn-e-Sina was famous for his original research in the field of medicine. He discovered the use of catheters. He gave intravenous injections by means of a silver syringe. He is famous because of his medical text book named, Al-Qanun-Fil-Tib (Qanoon). He also wrote Al -Shifa an encyclopaedia of philosophy. This comprises comprehensive treatise on logic, physics, mathematics and metaphysics.

1.4 PHYSICS AND SOCIETY

20th century is called the century of Physics and due to the

development in the field of science and technology we are justified to call it the modern scientific age. The development in science in general and in physics in particular made a great impact on our society. On tracing the history of civilization we find that the man used to live in caves (caveman) and he was afraid of lightning and thunder. With the development in the human knowledge the civilization also developed.

Machines were present much before the industrial developments. The motion of a lid of a kettle observed by George Stephenson enabled him to invent heat engine.

In 19th century the scientists were able to use electrical energy. The use of electric motors and generators accelerated the pace of industrialization. With further advances in the field of physics and technology, we are using micro wave ovens, refrigerators, air conditioners, vacuum cleaners, washing machines, etc. for our comfort and luxury. The invention of radio, television, telephone, video cassette recorder and others provided not only the means of recreation and luxury but also proved a mile stone in the field of communications and education.

The intelligent use of physics is observed in many fields of medicine and surgery e.g. from ordinary microscope to a sophisticated scanning electron microscope, the use of laser in surgery, the use of elaborate radiation system in nuclear medicine, the use of ultrasonic radiation for diagnostic purposes.

The impact of such developments and discoveries on our society has indeed been great and it is very likely that future discoveries and developments will be exciting challenging and of great benefit to humanity.

1.5 MEASUREMENT AND THE SYSTEM OF UNITS.

Quantitative physical measurements must be expressed by numerical comparison to some agreed set of standards. Thus all measurements are related to their chosen standards. The necessity for standards of various kinds has given rise to a number of measuring units.

In CGS the fundamental units of length, mass and time are centimetre, gram and second respectively.

In the British engineering system, the unit of force, length and time are chosen as the fundamental units. In it the unit of mass is a derived unit. The unit of force, length and time are pound, foot and second respectively.

In another system known MKS, the length, the mass and time are measured in metre, Kilogram and second respectively. In 1960, the genral conference of weights and measures, recommended that a metric system of measurement called International system of units abbreviated (MKSA) SI in all languages, system d'internationale be adopted. The SI units are derived from the earlier MKS system, so called because its first three basic units are metre (m), the kilogram (kg) and the second (s). These are expressed shortly.

1.6 MEASUREMENT OF TIME

Before 1960, the standard of time was defined in terms of mean solar day. Mean solar day is the length of a day measured throughout the year. A solar day is time interval between two successive appearance of the sun overhead. Thus mean second, representing the basic unit of time, was originally defined as (1/60) (1/60) (1/24) of a mean solar day. The time that is referred to rotation of the earth about its axis is called universal time. For reasons which we need not discuss, the length of the day varies throughout the year so that an average value has to be taken.

A high precision device for measuring with tremendously large accuracy is the atomic clock. The time can be measured to an accuracy of one part in 10^{12}. This is equivalent to an uncertainty of less than one second every 30000 years. The atomic clock is too complex a device to be described in detail, but briefly it is a radio transmitter giving out short waves (about 3 cm long), the frequency which is controlled by energy changes in gaseous caesium atoms. The great advantage here is the frequency (i.e the number of vibrations per second) of the changes is constant and not subjected to error. With this much high accuracy of atomic clock, the second was redefined in 1967, as the time interval occupied by 9192 631 770

cycles of radiation coresponding to a specified energy change in the caesium atom.

In addition to the basic unit of length, mass and time there are ampere, Kelvin, candela and mole in SI. For the sake of comprehension all are given below with brief description.

1.7 BASIC SI UNITS

(a) Time - Second (s)

A second is the duration of 9192631 770 periods of radiation corresponding to the transition between two hyperfine levels of the ground state of caesium-133 atom.

(b) Electric Current - ampere (A)

Ampere is the current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between the conductors a force equal to 2×10^{-7} newton per metre of length.

(c) Thermodynamic Temperature - Kelvin (K)

Kelvin, the unit of thermodynamic temperature, is 1/273.16 of the thermodynamic temperature of the triple point of water.

(d) Luminous Intensity - Candela (cd)

Candela is the luminous intensity, in the perpendicular direction of a surface 1/6 000 00 square metre of a black body at the temperature of freezing platinum under a pressure of 101325 newton per square metre.

(e) Amount of substance - mole (mol)

Mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.

1.8 Dimension

The word dimension has special meaning in physics. It is used to denote the nature of a physical quantity. Whether a distance is measured in any units, metres, miles or even light year, it

is always a distance and its dimension is length.

The symbols L, M and T are the symbols usually used to specify the dimensions of length, mass and time respectively. The dimension of any physical quantity can always be expressed as some combination of the fundamental quantities, such as mass, length and time.

The dimensions of velocity are written as L/T and the dimensions of area are L^2. The table 1.1 shows the dimension of some physical quantities.

Table 1.1

No.	Quantity	Dimensions
1	Area	L^2
2	Volume	L^3
3	Velocity	$L/T = LT^{-1}$
4	Acceleration	$L/T^2 = LT^{-2}$

In any equation the dimensions of a physical quantity must be same on both sides of equation. The dimensional tables attached to the various quantities may treated like algebraic quantities and may be combined, cancelled etc. Just as if they were factors in the equation.

Example:1.1

$$S = \vartheta_i t + \frac{1}{2} a t^2$$

The dimension of various physical quantities involved in the above are

ϑ_i is initial velocity, a is acceleration and t is time

(i) S ———— L

(ii) ϑ_i ———— $\frac{L}{T}$

(iii) a ———— $\frac{L}{T^2}$

Substituting we get

$$L = \frac{L}{T} \times T + \frac{1}{2}\ \frac{L}{T^2} \times T^2$$

$$L = L$$

Length = Length

The equation is dimensionally correct.

1.9 SIGNIFICANT FIGURES

All measurements of physical quantities involve some degree of in accuracy in them due to instrumental error and human error, etc and therefore the knowledge of precision of a measurement is very important. It is essential that we understand the limitations which experimental accuracy places on numbers that we measure. Let us now investigate how we handle numbers that are not exact. Suppose that the length of an object is recorded as 16.7 cm. This measurement is an approximation to the nearest length of a centimetre and its exact value lies between 16.65 and 16.75 cm. If this measurement is exact to the hundreth of a centimetre, it would have been recorded as 16.70 cm. The value 16.7 represents three significant figures (1, 6, 7), while the other value 16.70 represents four significant figures (1 6, 7, 0).

Thus a significant figure is the one which is known to be reasonably reliable. Similarly, a recorded mass of 6408.2 gm means that the mass was determined to the nearest tenth of a gram and represents five significant figures (6, 4, 0, 8, 2); the last figure 2, being reasonably correct guarantees the certainty of the preceding four figures.

Zeros may be significant, or they may merely serve to locate the decimal point for instance, the statement, that a truck has a weight 2500N, does not indicate definitely the accuracy of weighing. If it was weighed to the nearest 100N, the weight contains only two significant figures (2,5) and may be written as 2.5×10^3N. If the truck was weighed to the nearest 10N, the first zero is significant, but the second is not, the weight could be written 2.50×10^3N, displaying three significant figures. If it was weighed to the nearest

1N, the weight could be written as 2.500×10^3 N, four significant figures. If a zero stands between two significant figures, it is itself significant.

If a measured value is 8.3867, only three of whose digits are significant, we round it off to 8.39. A number is rounded off to the desired number of significant figures, by dropping one or more digits to the right. When the first digit dropped is less than 5, the last digit retained should remain unchanged. When the first digit dropped is more than 5 or when it is followed by digits not ab zeros, the last digit retained should be increased by 1.

If two numbers are divided or multiplied, the result has the same number of significant figures as the less accurate number. For example, the two numbers 4.71 and 5.642 are multiplied, the result is 26.6 and not 26.57382. Here we can claim three significant figures only since the less accurate number is, 4.71, contains three significant figures.

Problems.

1. Find the area of a rectangular plate having length (21.3 ± 0.2) cm and width (9.80 ± 0.10)cm.

   Ans (209 ± 4) cm^2

2. Calculate (a) the circumference of a circle of radius 3.5 cm and (b) area of a circle of radius 4.65 cm

3. Show that the expression $S=V_it + \frac{1}{2}at^2$ is dimensionally correct, when S is a co-ordinate and has unit of length, v_i is velocity, a is acceleration, and t is time.

4. Suppose the displacement of a particle is related to a time according to expression $S = ct^3$. What are the dimensions of the constant c.

5. Estimate the number of litres of gasoline used by all Pakistan's car each year

   given:

   No of cars in Pakistan = 500000. Average distance travelled per year by each car =16000 km gasoline consumption 6km/litre.

   Ans (13.33×10^8 litre)

CHAPTER: 2

Scalars and Vectors

2.1 SCALARS

Quantities which can be specified by a number having appropriate units (positive, negative, zero) are called scalars. For example, quantities such as temperature, density, volume, etc are scalars. The number representing any scalars is known as its magnitude. The scalars can be compared only when they have the same physical dimensions (units).

Two or more than two scalars measured in the same system of units are equal only if they have the same magnitude (absolute value) and sign. The scalars are denoted by letters in ordinary type. Operations, with scalars such as, division, subtraction, addition and multiplication follow the rules of elementary algebra.

2.2 VECTORS

Physical quantities having both magnitude and direction with appropriate unit are called vectors. For example, displacement, velocity, acceleration, force, moment of force, electrical field strength, are all vectors, because none of these quantities have a complete meaning without a mention of the direction.

A vector is represented graphically (Fig. 2.1) by a directed line segment or an arrow-head line segment, $\overrightarrow{QP}$, whose length and direction coincide with the magnitude and direction of the quantity under consideration respectively. The tail end-Q is regarded as initial point of the vector and the head-P is called terminal point of the vector.

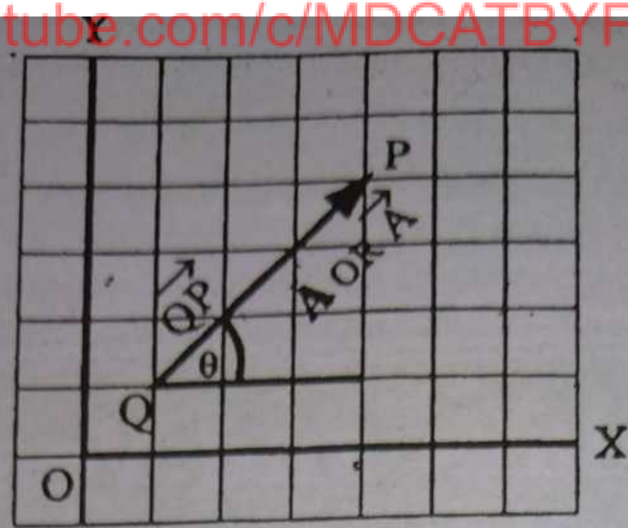


Fig. 2.1 θ - determines direction of the vector $\overrightarrow{QP}$ w.r.t.x-axis

Vectors are denoted by bold faced letters **A, B, C** and their magnitudes are denoted by **|A|, |B|, |C|** called the absolute value of **A, B, C**, respectively; more frequently we represent the magnitude alone by the italic letters, symbol, such as *A*, *B*, *C* respectively. In hand writing, it is convenient to put an arrow above the corresponding letters as $\vec{A}$, $\vec{B}$, $\vec{C}$ and their magnitudes are denoted by *A*, *B*, *C* respectively.

The following definitions are fundamental:

(a) Two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are equal if they have the same magnitude and similar direction without any consideration of the position of their initial points. fig 2.2(a). Thus

$\overrightarrow{OA} = \overrightarrow{OB}$ when (i) $OA = OB$

(ii) direction of $\overrightarrow{OA}$ is similar to the direction of $\overrightarrow{OB}$.

(b) A vector $-\overrightarrow{OA}$ represents a vector $\overrightarrow{OA}$ with opposite direction i.e the terminal point of vector $\vec{A}$ becomes its initial point and its initial point becomes the terminal point while the magnitude remains same as shown in Fig. 2.2.(b).

magnitude

also

$$\overrightarrow{OA} + (-\overrightarrow{OA}) = 0$$

The magnitude of a vector is always treated as non negative and the minus sign indicates the reversal of that vector through an

angle of 180°.

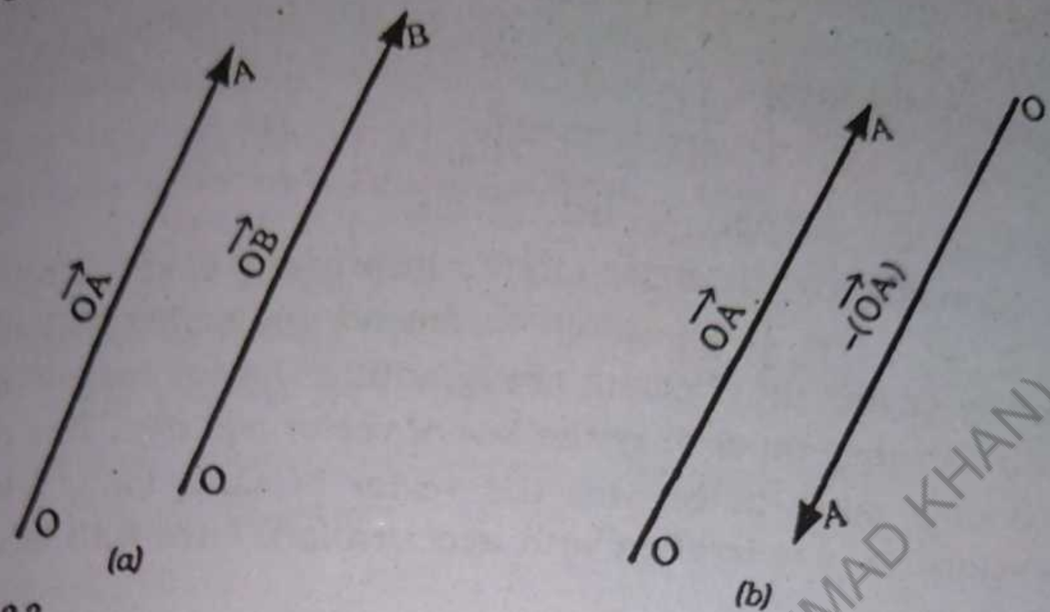


Fig. 2.2

2.3 ADDITION OF VECTORS

Consider two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ starting at a common point O as shown in Fig 2.3. Let these two vectors be the two adjacent sides of a parallelogram, complete the parallelogram OBCA and draw the diagonal OC. Assigning the direction by an arrow head to BC and AC similar to that of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively we get

$$\overrightarrow{BC} = \overrightarrow{OA}$$
$$\overrightarrow{AC} = \overrightarrow{OB}$$

By Definition the sum or resultant of the vectors $\overrightarrow{OB}$ and $\overrightarrow{BC}$ ($\overrightarrow{BC} = \overrightarrow{OA}$) is given by a vector $\overrightarrow{OC}$. (The diagonal of the parallelogram). This is the parallelogram law of vector addition. It is formed by placing the initial point of $\overrightarrow{BC}$ on the terminal point of $\overrightarrow{OB}$ and then joining the terminal point of $\overrightarrow{BC}$ to the initial point of $\overrightarrow{OB}$. The point O is then regarded as the initial point and the point C is regarded as the terminal point of the resultant vector. The direction of the resultant vector, $\overrightarrow{OC}$, is then from the initial point of $\overrightarrow{OB}$ (i.e. the point O) to the terminal point of $\overrightarrow{BC}$ (i.e the point C) as shown in Fig. 2.3 This method of vector addition is known as Head-to-tail rule and can be extended to accomplish ad-

dition of any number of vectors. This is also known as triangle law of vector addition.

consequently,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \qquad 2.1\ (a)$$

and $$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} \qquad 2.1\ (b)$$

Considering the above important property of vector combination, we are now in a position to amend our earlier definition of vector by saying that vectors are quantities having magnitude, direction and they must obey the law of vector addition. This combination only takes place with the vector of same kind, velocities with velocities, acceleration with acceleration, force with force and so on.

Analytical determination of resultant of two vectors and its direction.

In fig 2.3 consider the triangle OCA. Representing OA, AC and OC by A, B and R respectively, we have by the law of cosines

$$R^2 = A^2 + B^2 - 2AB \cos \angle OAC \qquad 2.2\ (a)$$

$$R = \sqrt{A^2 + B^2 - 2AB \cos \angle OAC} \qquad 2.2\ (b)$$

The Eq. 2.2(b) determines the magnitude, R, of the result vector $\vec{R}$ by the law of sines.

$$\frac{A}{\sin\angle ACO} = \frac{B}{\sin\angle AOC} = \frac{R}{\sin\angle OAC} \qquad 2.3$$

The Eq: 2.3 determines the direction of the resultant vector, $\vec{R}$.

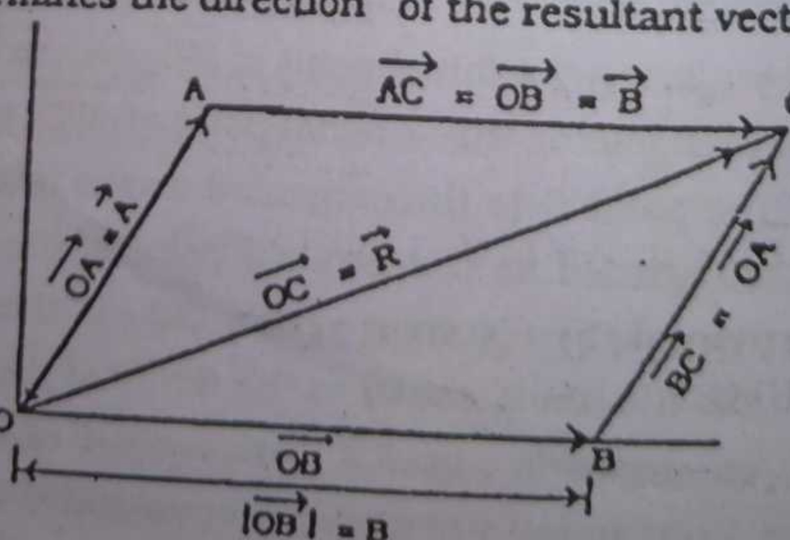


Fig: 2.3

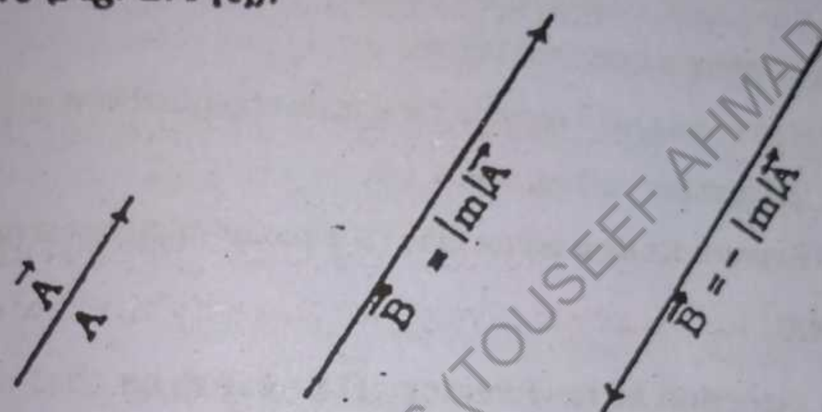
2.4 MULTIPLICATION OF A VECTOR BY A NUMBER

The operation of multiplication of a vector by a number is simple and straight forward. The product of number m, and a vector A as shown in Fig. 2.4 (a), generates a new vector, say $\vec{B}$, whose magnitude is $|m|$ times the magnitude of vector $\vec{A}$, therefore,

$$B = |m| A \qquad 2.4$$

(i) The direction of vector $\vec{B}$, is same as that of vector $\vec{A}$ if m is + ve (Fig. 2.4 (b)).

(ii) The direction of vector $\vec{B}$ is opposite to that of vector $\vec{A}$ if m is - ve (Fig. 2.4 (c)).



(a) (b) (c)

Fig 2.4 (a) *Represents original vector*
(b) *Represents new vector after multiplication when scaler mulipler is positive*
(c) *Represents new vector after multiplication when scaler multiplier, m, is negative*

The multiplication of a vector by one or more number (say m,n) is governed by the following rules:

$$m\vec{A} = \vec{A}m \quad : \text{commutative law for multiplication} \qquad 2.5(a)$$

$$m(n\vec{A}) = (mn)\vec{A} \quad : \text{associative law for multiplication} \qquad 2.5(b)$$

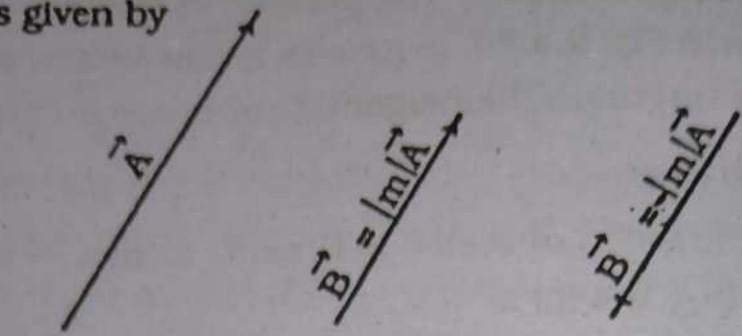
$$(m+n)\vec{A} = m\vec{A} + n\vec{A} \quad : \text{distributive law} \qquad 2.5(c)$$

$$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B} \quad : \text{distributive law} \qquad 2.5(d)$$

2.5 DIVISION OF A VECTOR BY A NUMBER (Non zero)

The division of a vector $\vec{A}$, by a number, n, is simple and involves the multiplication of the vector by the reciprocal of the num-

ber n, with the result a new vector is generated. Let n represents a number and its reciprocal $m = {}^1/_n$, then the magnitude of new vector (say $\vec{B}$) is given by



(a) (b) (c)

Fig: 2.5 (a) Represents original Vector

(b) Represents new vector after division by number n = 1/m or by number multiplier m = 1/n

(C) Represents new vector when the scalar multiplier is negative.

$\vec{B} = |m|A$

(i) The direction of new vector $\vec{B}$ is same as that of $\vec{A}$, if the number is + ve. Fig 2.5 (b)

(ii) The direction of new vector $\vec{B}$ is opposite to that of $\vec{A}$, if the multiplier is - ve, Fig. 2.5 (c).

2.6 UNIT VECTOR

A vector $\vec{A}$ in any given direction and whose magnitude is unity (A = 1) is referred as a unit vector. We use special notation $\hat{a}$

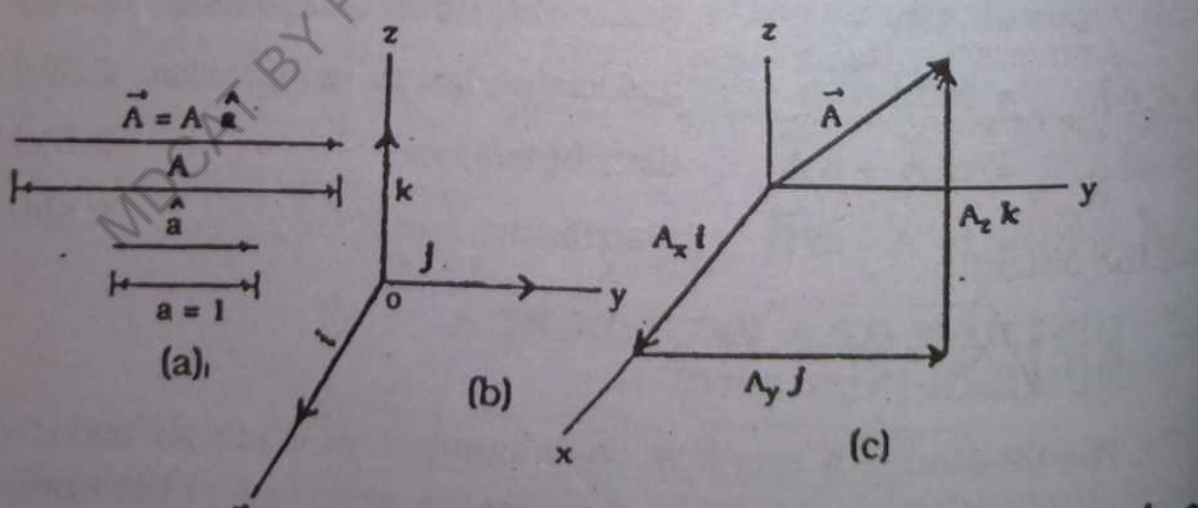


Fig. 2.6 a, b, c

(read as 'a hat') to represent the unit vector. A unit vector can be obtained by dividing the vector by its magnitude i.e.

$$\frac{\vec{A}}{A} = \hat{a} \qquad 2.6\ (a)$$

unit vector only specifies the direction of a given vector.

also $\vec{A} = A\ \hat{a}$ 2.6(b)

The vector $\vec{A}$ which has magnitude A, is just A times the unit vector $\hat{a}$ and has the same direction as $\hat{a}$ as shown in Fig:2.6 (a).

An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three dimensional rectangular coordinate system, and are denoted by i, j and k respectively as shown in figure 2.6 (b). These are called rectangular unit vectors.

Let A_x, A_y and A_z be the rectangular coordinates of the terminal point of a vector $\vec{A}$ with its initial point placed at the origin of a rectangular coordinate system as shown in Fig. 2.6 (c). Then by definition [Eq 2.6(b)] the vectors $|\vec{A}_x|i$, $|\vec{A}_y|j$ and $|\vec{A}_z|k$ are referred as the rectangular component vectors of the vector $\vec{A}$ in the direction of positive x.y and z axes respectively. Also $|\vec{Ax}|$, $|\vec{Ay}|$ and $|\vec{Az}|$ are called rectangular components of $\vec{A}$ along positive x.y and z axes respectively.

Conversely, the sum of rectangular components vectors produces the original vector $\vec{A}$, i.e

$$\vec{A} = A_x i + A_y j + A_z k \qquad 2.7$$

and the magnitude of $\vec{A}$ is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad 2.8$$

Here we take $i^2 = j^2 = k^2 = 1$ This shall be explain when we deal with the multiplication of 0 vector by a vector.

Example 2.1

Find the unit vector parallel to the vector.

$$\vec{A} = 3i + 6j - 2k$$

Solution

Using Eq. 2.8 the magnitude of vector $\vec{A}$ is given by

$$|\vec{A}| = \sqrt{(3)^2 + (6)^2 + (-2)^2} = 7$$

then the unit vector parallel to $\vec{A}$ is given by Eq.2.6(a)

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{3i + 6j - 2k}{7}$$

$$\hat{a} = \frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k$$

By definition the magnitude of unit vector is 1 and therefore we can check our result by evaluating the magnitude of unit vector, i.e

$$\left|\frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k\right| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{-2}{7}\right)^2} = 1$$

2.7 FREE VECTOR

A vector such as the velocity of a body undergoing uniform translational motion, which can be displaced parallel to itself and applied at any point, is known as a FREE VECTOR Fig.2.7. It can be specified by giving its magnitude and any two of the angles between the vector and the coordinate axes. In three dimensions a free vector is uniquely determined by its three projections on the axes of a rectangular coordinate system.

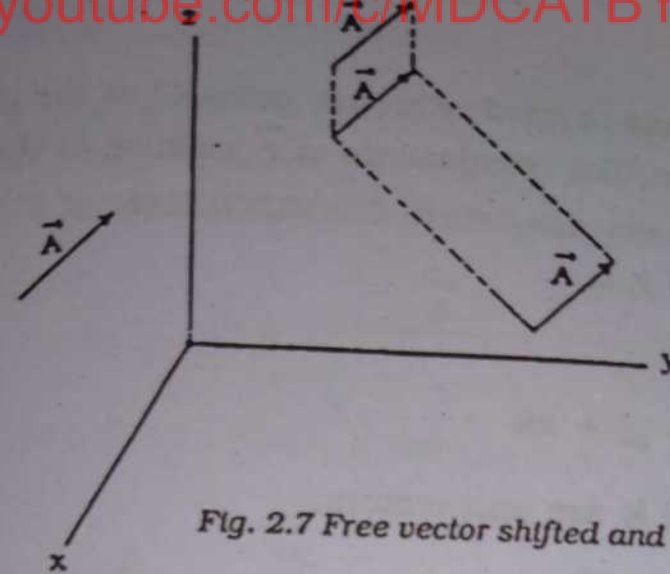


Fig. 2.7 Free vector shifted and drawn parallel to itself

2.8 POSITION VECTOR

Suppose we have a fixed reference point O, then we can specify the position of a given point P w.r. to the point O by means of a vector having magnitude and direction represented by a directed line segment $\overrightarrow{OP}$ as shown in Fig. 2.8(a). This vector is called position vector. We call $\overrightarrow{OP}$ a position vector, since it determines the position of the point P relative to the fixed point O.

Let $\vec{r}$ be a position vector of a point P relative to a rectangular coordinate system defined by unit vectors $\boldsymbol{i}, \boldsymbol{j}, \mathbf{k}$ and starting at

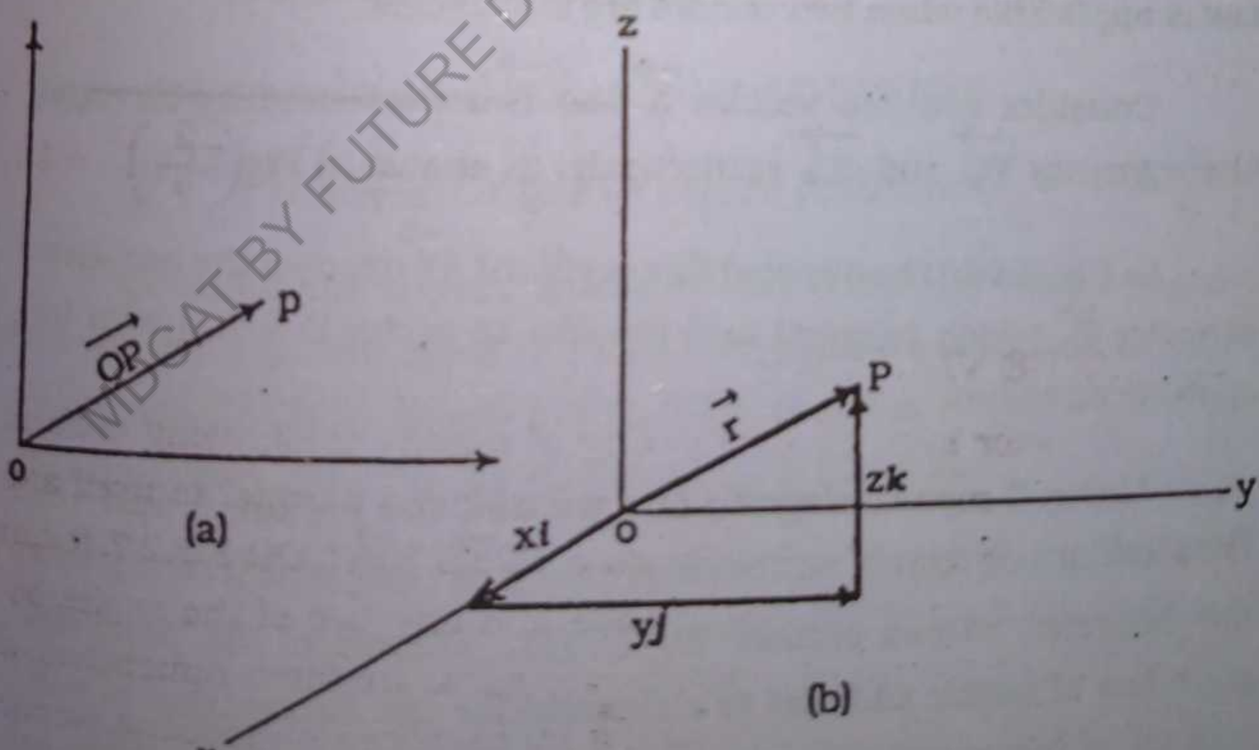


Fig. 2.8 (a) A position vector.
(b) A position vector in rectangular coordinate system.

the origin (corresponds to fixed reference point O) of the rectangular coordinate system. The components of $\vec{r}$ relative to the rectangular coordinate system are called COORDINATES of P and are usually denoted by x, y, z.

Thus

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{2.9}$$

Where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors.

The magnitude of $\vec{r}$ is given by

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \tag{2.10}$$

It is important to note that while all vectors have components, only the components of the position vectors are known as coordinates.

2.9 NULL VECTOR

We have seen that the vectors combine or add according to the parallelogram law. We would like to examine whether the same law is applicable when two vectors are subtracted.

Consider two free vectors $\vec{A}$ and $\vec{B}$ represented by directed line segments $\vec{PQ}$ and $\vec{RS}$ respectively, as shown in Fig.2.9(a).

In Fig. 2.9(b) the directed line segment $\vec{XY}$ denotes the negative of vector $\vec{B}$, which is equal and parallel to vector $\vec{B}$ but drawn in opposite direction.

Using the parallelogram law we add the vectors $\vec{A}$ and $-\vec{B}$. The resultant or sum of vectors is given by $\vec{A} + (-\vec{B})$ which represents the difference of two vectors Fig 2.9 (c). Therefore, the parallelogram law of vector addition is also valid for the subtraction of vectors i.e, if two vectors are identical in magnitude and opposite in direction, then difference vector $\vec{A} + (-\vec{B})$ is called NULL or ZERO

vector. The null vector has zero magnitude and has no direction or it may have any direction. Nevertheless we shall accept it as vector though it really does not quite fit to our definition of a vector.

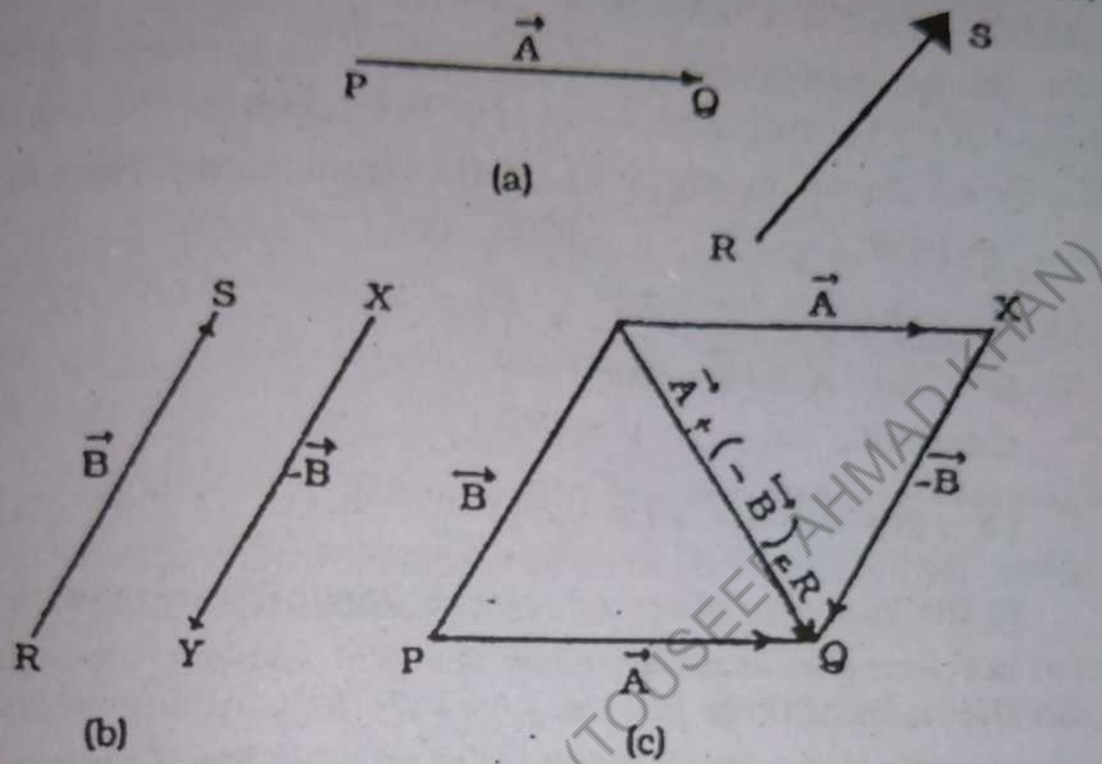


Fig.2.9 (a) Two free Vectors $\vec{A}$ and $\vec{B}$
(b) Negative of Vector $-\vec{B}$
(c) Difference Vector $\vec{A}+(-\vec{B})$

2.10 PROPERTIES OF VECTOR ADDITION

(i) *Commutative law of vector addition.*

Consider two vectors $\vec{A}$ and $\vec{B}$. Let these two vectors represent the two adjacent sides of a parallelogram. We construct the parallelogram OACB as shown in Fig. 2.10, then the diagonal OC represents the resultant vector $\vec{R}$. From Fig. 2.10 we have

$$\vec{R} = \vec{A} + \vec{B} \qquad 2.11\ (a)$$

$$\vec{R} = \vec{B} + \vec{A} \qquad 2.11\ (b)$$

therefore

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \qquad 2.12$$

In the language of vector algebra, this fact is referred as the commutative law of vector addition.

(ii) Associative law of vector addition.

Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, as shown in Fig. 2.11, using HEAD-TO-TAIL rule, we obtain the resultant $(\vec{A}+\vec{B})$ and $(\vec{B}+\vec{C})$ as shown in Fig 2.11. Once again using Head-to-tail rule, we write

$$\vec{R} = (\vec{A}+\vec{B})+\vec{C} \qquad 2.13$$

$$\vec{R} = \vec{A}+(\vec{B}+\vec{C}) \qquad 2.14$$

therefore

$$(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C}) \qquad 2.15$$

In the language of vector algebra, this property of vector addition is referred as associative law of vector addition. Consequently, on the basis of these laws we conclude that the sum of vectors remains same irrespective of any order or grouping of vectors.

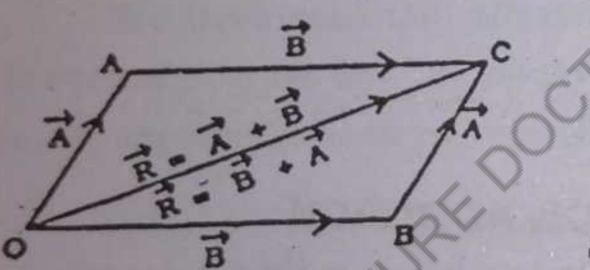


Fig. 2.10
Commutative law of vector addition

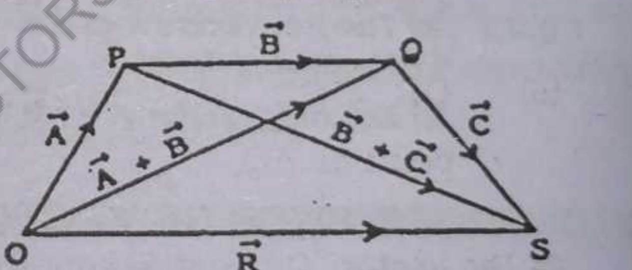


Fig 2.11
Associative law of vector addition

Example 2.2

Represent graphically three displacement vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ having magnitudes OA, OB, OC and making angles of 0°, 40°, 70° with respect to positive x-axis (measured counter clockwise) respectively. Determine the resultant displacement vector and its direction w.r.t. x-axis

Solution

Choosing the appropriate scale of magnitude, the required

vectors are drawn as shown in Fig. 2.12(a), Fig. 2.12(b) and Fig 2.12(c). For the determination of the resultant displacement vector we proceed as follows:

Step 1. First form the resultant displacement vector $\overrightarrow{OQ}$ by combining $\overrightarrow{OA}$ and $\overrightarrow{OB}$, according to the parollelogram law as shown in Fig 2.12(d). The magnitude and direction of $\overrightarrow{OQ}$ can easily be measured.

Step 2. Combine the vector $\overrightarrow{OQ}$ and $\overrightarrow{OC}$ by the law of parallelogram and obtain the resultant displacement vector $\overrightarrow{OR}$ as shown in Fig. 2.12(e). The magnitude and direction of $\overrightarrow{OR}$ can easily be measured.

Alternatively, we can obtain the resultant displacement vector $\overrightarrow{OR}$ by assuming all vectors as free vectors and simply using the Head-to-tail rule as shown in Fig. 2.12 (f) observe that

(i) $\overrightarrow{AQ} = \overrightarrow{OB}$, whose initial point lies on the terminal point of $\overrightarrow{OA}$

(ii) $\overrightarrow{QR} = \overrightarrow{OC}$, whose initial point lies on the terminal point of $\overrightarrow{AQ}$

As the vector $\overrightarrow{OR}$ represents the resultant vector, its magnitude $|\overrightarrow{OR}|$ is given by its length and the direction is given by $\angle ROA$.

From Fig.2.12(f), we observe

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$$OR \neq OA + OB + OC$$

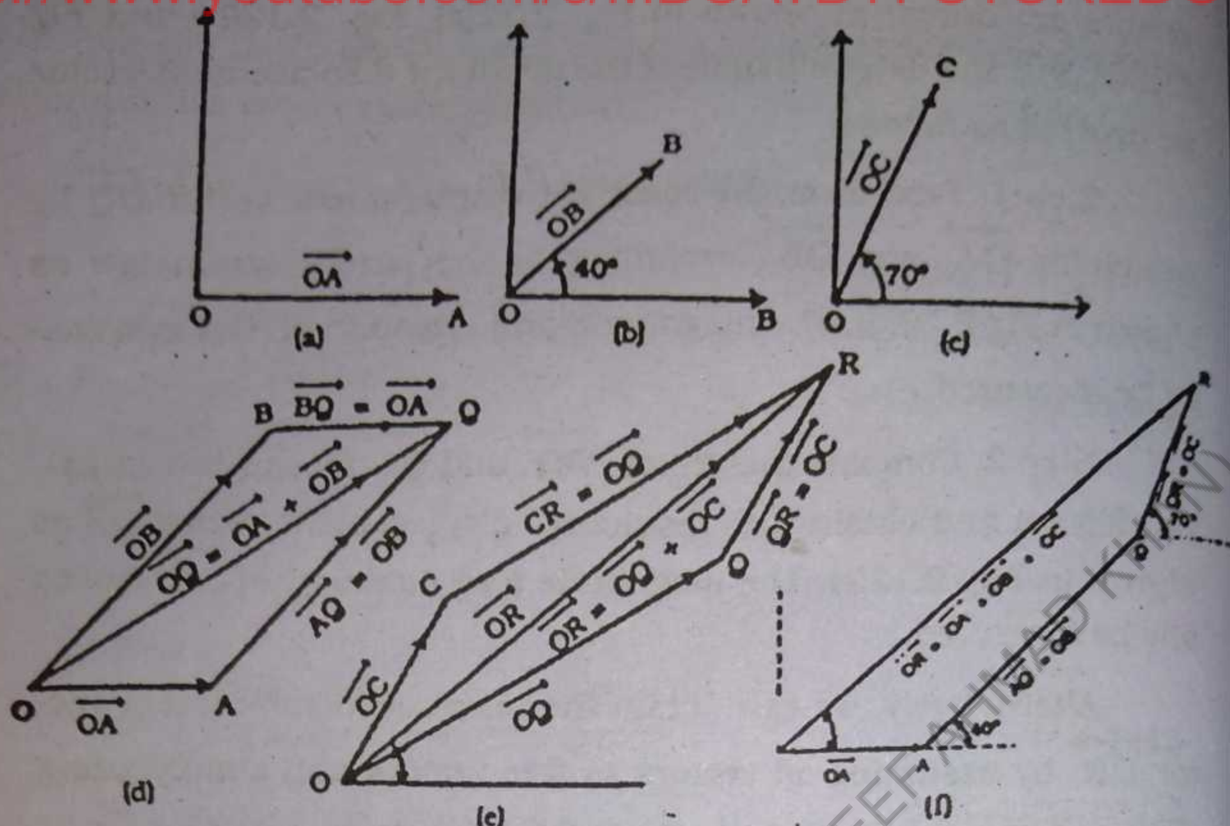


Fig. 2.12

(a) A directed line segment $\overrightarrow{OA}$

(b) A directed line segment $\overrightarrow{OB}$

(c) A directed line segment $\overrightarrow{OC}$

(d) Sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

(e) Sum of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(f) Sum of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ obtained by applying Head-to-tail rule.

Example 2.3

A point P is subjected to six different forces such as $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$, $\overrightarrow{F_4}$, $\overrightarrow{F_5}$, $\overrightarrow{F_6}$ as shown in Fig. 2.13 (a). Find out the resultant force at the point P.

Solution

Assuming all vectors as free vectors, we begin by first drawing the vector $\overrightarrow{F_1}$ parallel to itself as shown in Fig 2.13 (b) Applying Head-to-tail rule, we place the initial point of vector $\overrightarrow{F_2}$ on the terminal point of $\overrightarrow{F_1}$ in such a way that the vector $\overrightarrow{F_2}$ remains parallel to itself. Then we place the initial point of vector $\overrightarrow{F_3}$ on the terminal point of vector $\overrightarrow{F_2}$ while maintaining the original direction and magnitude of the Vector $\overrightarrow{F_3}$ and so on, till the initial point of $\overrightarrow{F_6}$ is placed on the terminal point of $\overrightarrow{F_5}$. Finally, join the terminal point of vector $\overrightarrow{F_6}$ with the point P and this gives the resultant vec-

tor R which is equal to

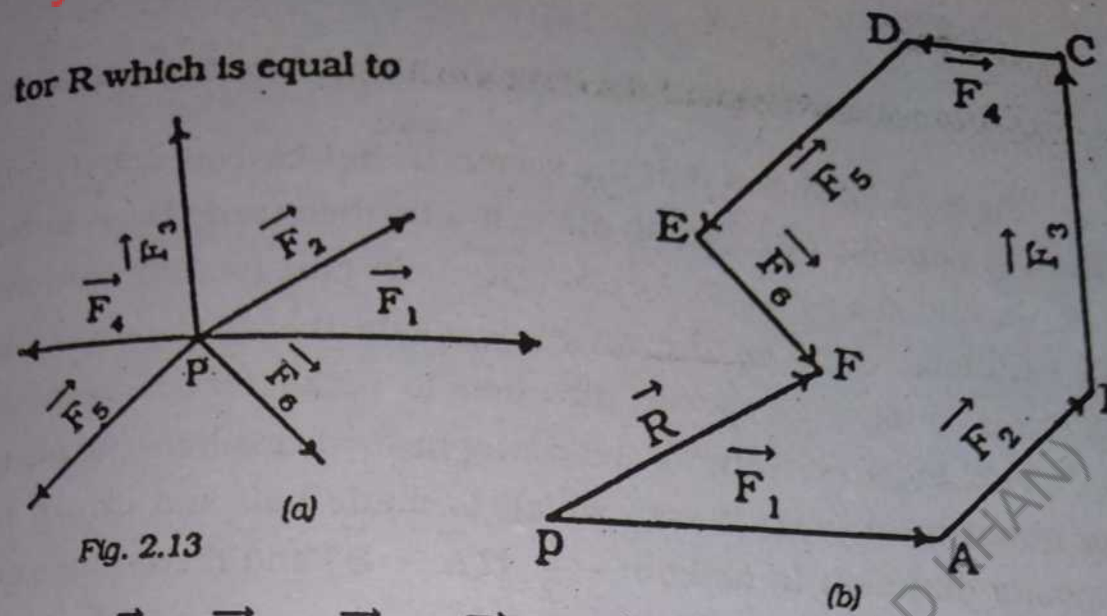


Fig. 2.13

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$$

The force equal and opposite to $\vec{R}$ (i.e $-\vec{R}$) when applied on the point P will prevent any displacement of the point P.

Example 2.4

Given three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in Fig.2.14(a), construct

(a) $\vec{A} - 2\vec{B} + 2\vec{C}$ (b) $4\vec{C} - \frac{1}{2}(2\vec{A} - \vec{B})$

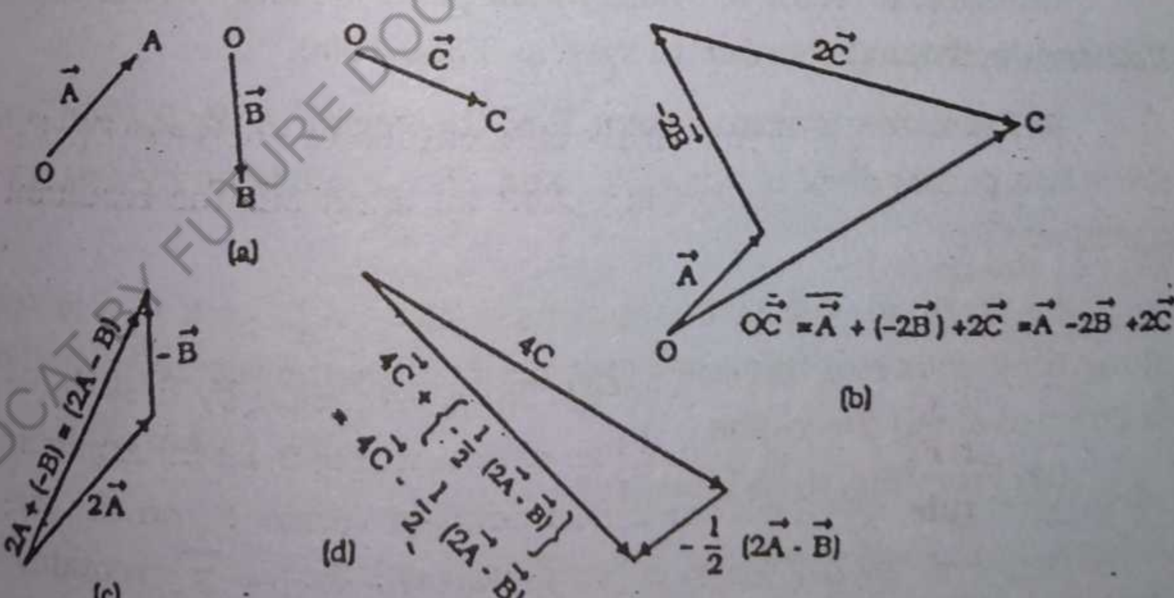


Fig. 2.14 (a) Vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ of given magnitudes and directions.

(b) Graphical construction to obtain $\vec{A} - 2\vec{B} + 2\vec{C}$.

(c) Graphical construction to obtain $2\vec{A} - \vec{B}$.

(d) Graphical construction to obtain $4\vec{C} - \frac{1}{2}(2\vec{A} - \vec{B})$

Solution

Fig.2.14 (b) shows that the vector $\overrightarrow{B}$ has become double and drawn in opposite direction to make it $-2\overrightarrow{B}$, then using Head-to-tail rule, we find the resultant vector $\overrightarrow{OC}$. In part (b), first we form the resultant vector of the quantity inside the parentheses in which $\overrightarrow{B}$ is drawn opposite direction to make $-\overrightarrow{B}$ as shown in Fig.2.14(c).Fig.2.14(d) shows the newly formed resultant vector of the quantity in Parentheses which is made half and drawn in opposite direction to make it $-\frac{1}{2}(2\overrightarrow{A}-\overrightarrow{B})$ and then combined with $4\overrightarrow{C}$ to form the resultant vector.

2.11 RESOLUTION AND COMPOSITION BY RECTANGULAR COMPONENTS

The graphical method already discussed for the addition of vectors is inconvenient for vectors defined in two or three dimensions. Keeping in view this situation we now discuss a method for addition of vector, which is analytical. Here the given vector is resolved into components w.r.to a particular coordinate system.

Consider a vector $\overrightarrow{A}$ whose initial point is placed at the origin of two dimensional coordinate system. Fig.2.15(a).

(i) From the terminal point P of the vector $\overrightarrow{A}$ fig.2.15(b), we draw two perpendicular lines PQ and PS on x-axis and y-axis, respectively .

(ii) The line OQ is denoted by vector . $\overrightarrow{A}_x$, as it is directed along the x-axis and the line PQ is denoted by the vector, $\overrightarrow{A}_y$, and it is directed along the y-axis.

(iii) From Fig. 2.15(b) we see.

$$\overrightarrow{A} = \overrightarrow{A}_x + \overrightarrow{A}_y$$

The vectors $\overrightarrow{A}_x$ and $\overrightarrow{A}_y$ are referred, as rectangular vector components.

From the fig.2.15 (c)

$$A_x = A\cos\theta \qquad 2.16\,(a)$$

$$A_y = A\sin\theta \qquad 2.16\,(b)$$

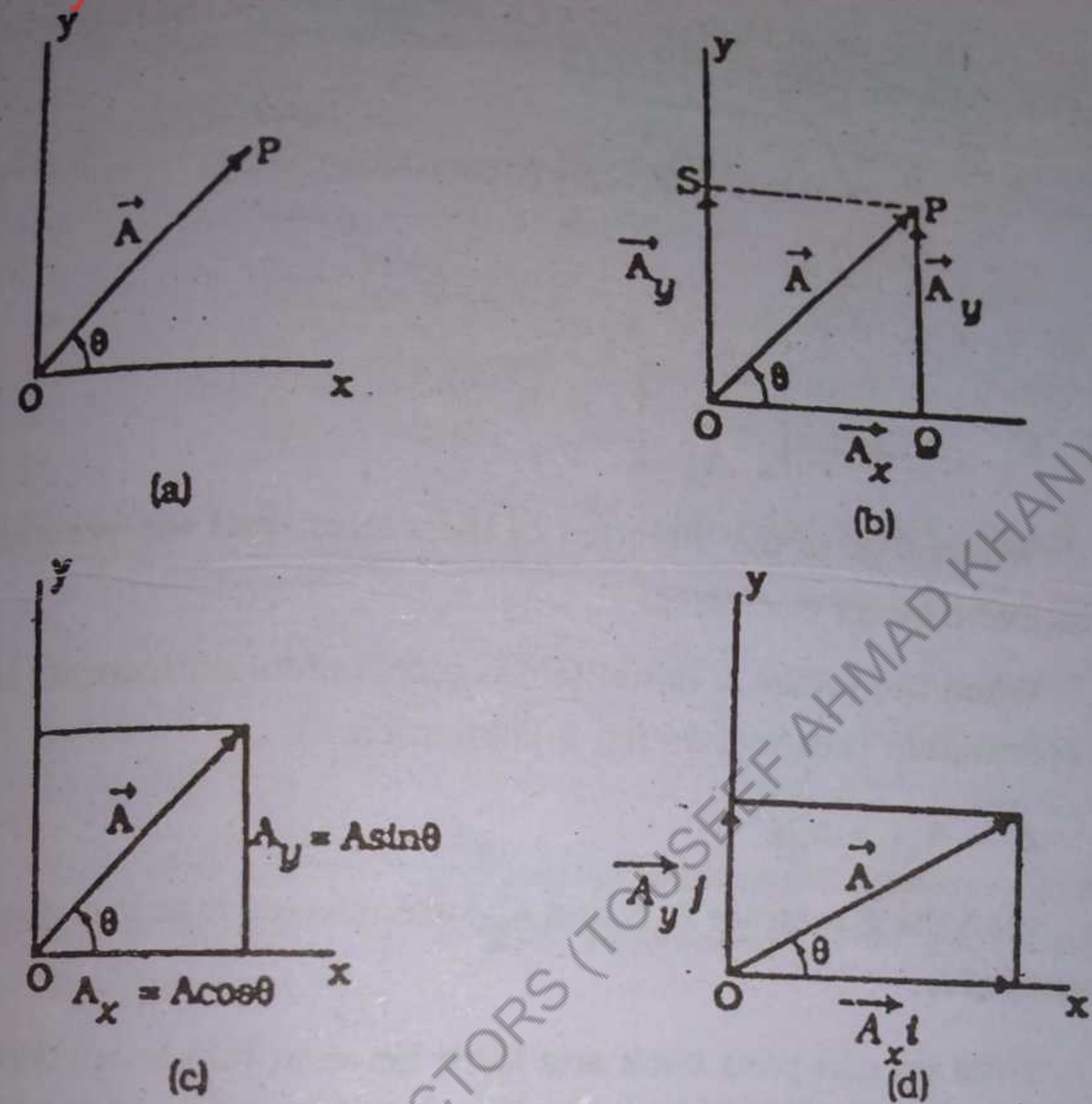


Fig. 2.15 (a) A vector $\vec{A}$ and perpendicular x and y axes
(b) The components of $\vec{A}$ are $\vec{A}_x$ and $\vec{A}_y$
(c) Resolution of vector $\vec{A}$ into its scalar components
(d) Resolution of vector $\vec{A}$ into its vector components

where A, A_x and A_y represent the magnitudes of $\vec{A}$, $\vec{A}_x$ and $\vec{A}_y$ respectively.

Once a vector is resolved into its rectangular components, the components are then used to specify the vector. These components of a vector behave like scalar quantities as depicited in Fig. 2.15 (c)

Conversely, we can obtain the orginal vector once its components are known. That is, we can obtain the magnitude of the vector and its direction from the knowledge of its components. The process by which a vector can be reconstituted from its compo-

nents is know as composition of a vector. To obtain magnitude and dirction, we refer to Fig. 2.15 (b)

$$A = \sqrt{A_x^2 + A_y^2} \quad 2.17$$

$$\tan\theta = \frac{A_y}{A_x} \quad 2.18$$

$$\theta = \tan^{-1}\left[\frac{A_y}{A_x}\right] \quad 2.19$$

Where θ gives the direction of the vector w.r.t the +ve x-axis measured counter clockwise.

When the vector $\vec{A}$ is written in terms of its components and the rectangular unit vectors fig. 2.15(d) such as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} \quad 2.20$$

then the quantities $A_x\hat{i}$ and $A_y\hat{j}$ are referred to as vector components of $\vec{A}$.

Thus we can pass back and forth between the description of a vector in terms of its magnitude A and direction θ and the quivalent description in terms of its components.

Having dealt with the resolution and the composition of a vector, we now turn toward its application. The resolution and composition of a vector provide an analytical tool for addition of any number of vectors in a given coordinate system. Once again we restrict our discussion to two-dimensional coordinate system. The method follows as under:

Step: 1 Resolve each given vector into its rectangular components i.e, x-component and y-component.

Step 2. Find the algebraic sum of all the individual x-components, the sum then represents the component of the sum vector along x-axis.

Step: 3 Find the algebraic sum of all the individual y-components, the sum then represents the component of the sum vector along y-axis.

Step: 4 Find the magnitude of the sum vector or of the resultant vector by Eq.No.2.17

Step: 5. Find the direction (i.e the value of angle θ w.r.t. +ve x-axis measured counter clockwise of the resultant vector.

2.12 ADDITION OF VECTORS BY RECTANGULAR COMPONENTS

Consider two vectors $\vec{A}_1$ and $\vec{A}_2$, having magnitude A_1 and A_2 respectively. The vector $\vec{A}_1$ makes an angle θ_1 and the vector $\vec{A}_2$ makes an angle θ_2 with the + ve x-axis as shown in Fig 2.16 (a).

(i) Resolve the vector $\vec{A}_1$ into its rectangular components $\vec{A}_{1x}$ and $\vec{A}_{1y}$ as shown in Fig. 2.16(b). The magnitude of these component vectors is given by

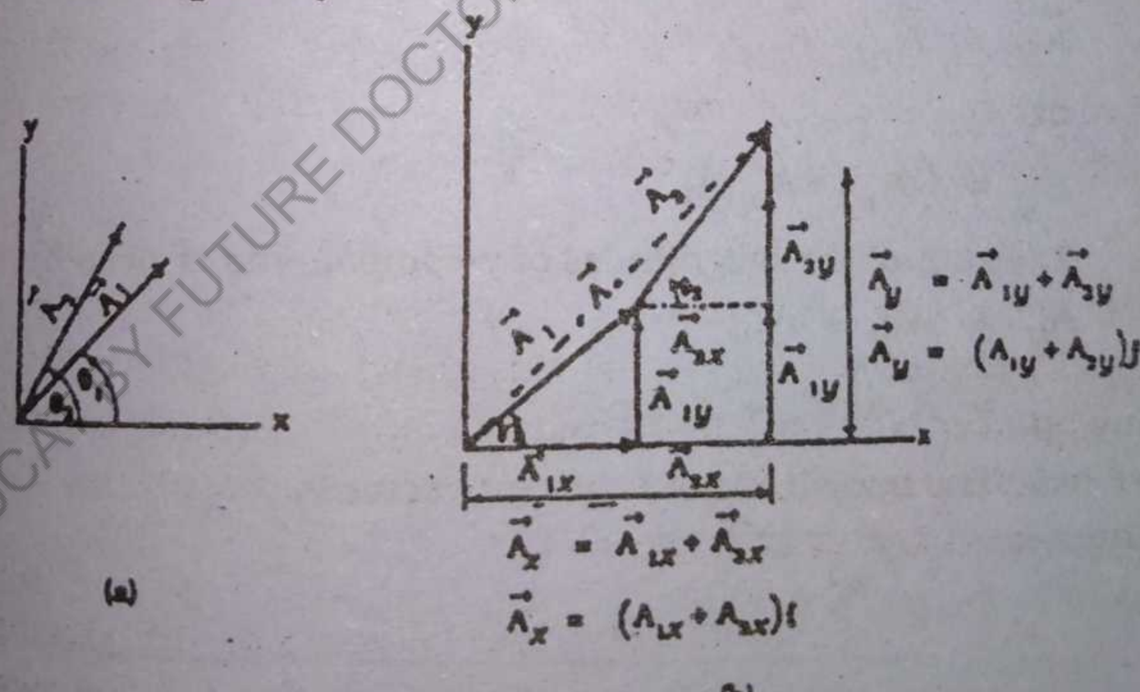


Fig 2.16 (a) Vectors $\vec{A}_1$ and $\vec{A}_2$ and perpendicular x and y axes.

(b) Resolution of vectors $\vec{A}_1$ and $\vec{A}_2$ into their components.

$$A_{1x} = A_1\cos\theta_1 \qquad 2.21\ (a)$$

$$A_{1y} = A_1\sin\theta_1 \qquad 2.21\ (b)$$

(ii) Move the vector $\vec{A}_2$ parallel to itself, so that its initial point lies on the terminal point of vector $\vec{A}_1$ as shown in Fig. 2.16(b).

(iii) Resolve the vector $\vec{A}_2$ into its rectangular components $\vec{A}_{2x}$ and $\vec{A}_{2y}$ as shown in Fig. 2.16(b) then magnitude of each component is given by

$$A_{2x} = A_2\cos\theta_2 \qquad 2.22\ (a)$$

$$A_{2y} = A_2\sin\theta_2 \qquad 2.22\ (b)$$

(iv) The resultant vector along x-axis is given by the algebraic sum of the component vectors along x-axis

$$\vec{A}_x = (\vec{A}_{1x} + \vec{A}_{2x})\,i \qquad 2.23$$

The sum of the magnitudes of x-components is given by

$$A_x = A_{1x} + A_{2x} \qquad 2.24(a)$$

$$A_x = A_1\cos\theta_1 + A_2\cos\theta_2 \qquad 2.24(b)$$

(v) Similarly the sum of component vectors along y-axis

$$\vec{A}_y = \vec{A}_{1y} + \vec{A}_{2y} \qquad 2.25$$

or

$$\vec{A}_y = \left(A_{1y} + A_{2y}\right)j$$

The sum of the magnitudes of y-components is given by

$$A_y = A_{1y} + A_{2y} \qquad 2.26(a)$$

$$A_y = A_1\sin\theta_1 + A_2\sin\theta_2 \qquad 2.26(b)$$

(vi) The magnitude of resultant vector is

$$A = \sqrt{A_x^2 + A_y^2} \qquad 2.27(a)$$

$$A = \sqrt{[A_1\cos\theta_1 + A_2\cos\theta_2]^2 + [A_1\sin\theta_1 + A_2\sin\theta_2]^2} \qquad 2.27(b)$$

The direction of resultant vector

$$\theta = \tan^{-1}\left[{A_y}/{A_x}\right] \qquad 2.28$$

Example 2.5

An automobile travels 200 km due east and then 150 km 60° north of east. Determine the resultant displacement and the direction of the resultant with respect to positive x-axis.

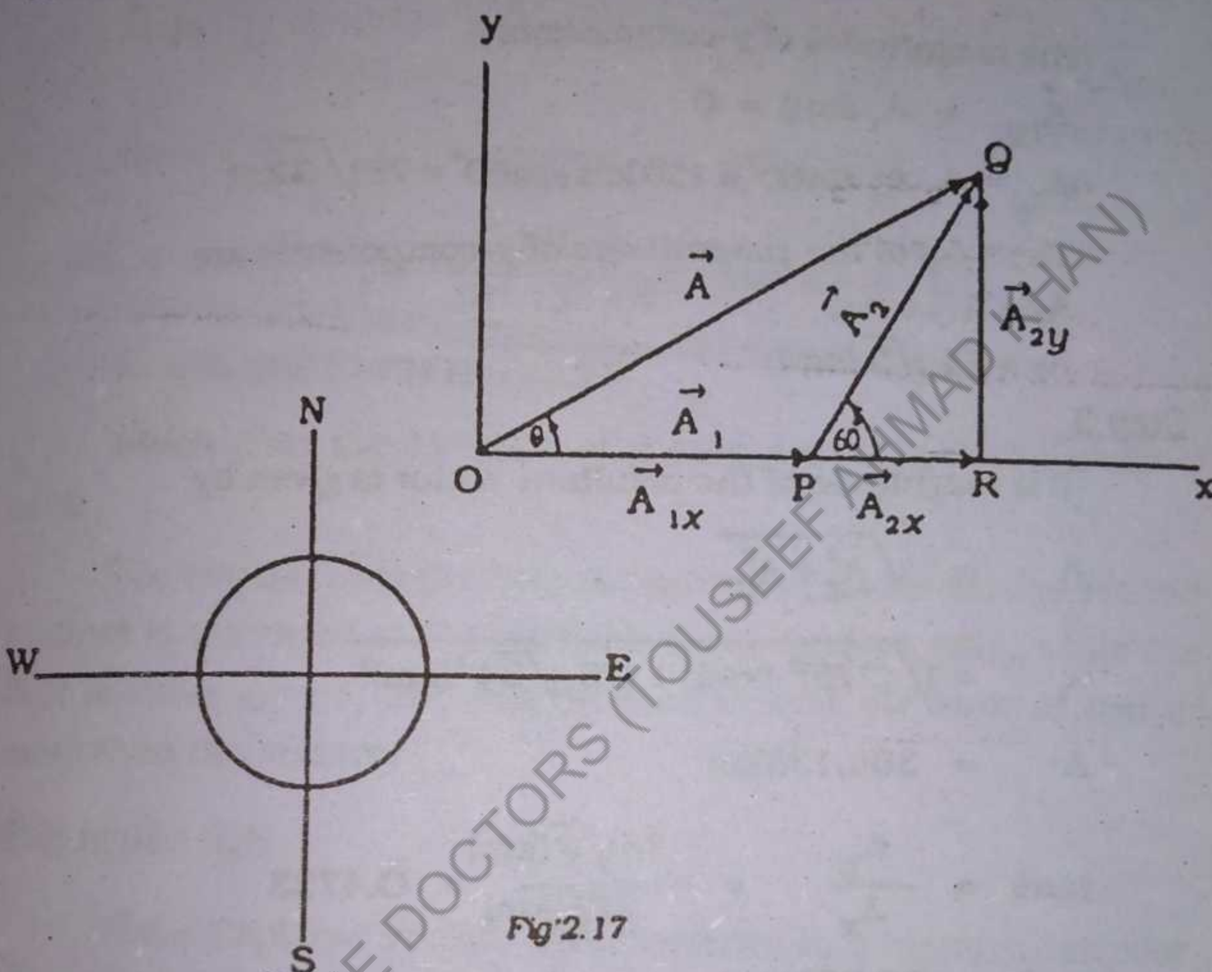


Fig 2.17

Solution

Method—1

We choose the positive x-axis to be east and positive y-axis to be north direction Fig 2.17. First we draw the line OP=4cm (50km=1cm)along the +ve x-axis which represents 200 km displacement vector $\vec{A}$. Next we draw the line PQ=3cm which makes and angle of 60° with +ve x-axis measured counter clockwise, this represents 150km displacement vector $\vec{A_2}$

Step 1. The magnitudes of x-components are

$$A_{1x} = A_1\cos 0 = 200\text{km} \quad [\because \theta = 0]$$

$$A_{2x} = A_2 \cos 60^\circ = 150\,\text{km}\cos 60^\circ = 75\text{km}$$

The sum of the magnitudes of x-components

$$A_x = A_{1x} + A_{2x}$$
$$= 275 \text{ km}$$

Step 2.

The magnitudes of y-components

$$A_{1y} = A_1 \sin\theta = 0$$
$$A_{2y} = A_2 \sin 60^\circ = 150 \text{km} \sin 60^\circ = 75\sqrt{3} \text{km}$$

The sum of the magnitudes of y-components are

$$A_y = A_{1y} + A_{2y}$$
$$A_y = 75\sqrt{3} \text{ (km)}$$

Step 3.

The magnitude of the resultant vector is given by

$$A = \sqrt{A_x^2 + A_y^2}$$
$$= \sqrt{(275)^2 (\text{km})^2 + (75\sqrt{3})^2 (\text{km})^2}$$
$$A = 304.138 \text{km}$$

$$\tan\theta = \frac{A_y}{A_x} = \frac{75\sqrt{3} \text{ (km)}}{275 \text{ (km)}} = 0.4723$$
$$\theta = 25.28^\circ$$

thus magnitude of resultant vector = 304.138 km and direction of the resultant vector is 25.28° north of east.

Method—2

Alternatively: from triangle OPQ, we have by the law of Cosines.

$$A = \sqrt{A_1^2 + A_2^2 \quad 2A_1A_2 \cos\angle OPQ}$$
$$A = \sqrt{(200)^2 (\text{km})^2 + (150)^2 (\text{km})^2 - 2 \times 200\text{km} \times 150 \text{ km} \cos 12}$$
$$A = 304.138 \text{km}$$

Also by the law of sines

$$\frac{A}{\sin\angle OPQ} = \frac{A_1}{\sin\angle PQO} = \frac{A_2}{\sin\angle QOP}$$

$$\frac{A_2}{\sin\angle QOP} = \frac{A}{\sin\angle OPQ}$$

$$\sin\angle QOP = \frac{A_2}{A} \quad \sin\angle OPQ$$

$$= \frac{150\text{km}}{304.138\text{ km}} \quad \sin 120^\circ = 0.4271$$

$$\therefore \theta = \sin^{-1}(0.4271) = 25.28^\circ$$

which gives the direction of the resultant i.e; 25.28° north of east.

The two methods produce same results. However, the second method is restricted to the addition of two vectors only, while the first is more general and can be used to find the sum of two or more than two vectors.

Example 2.6

Three Coplanar vectors with reference to a rectangular coordinate system are

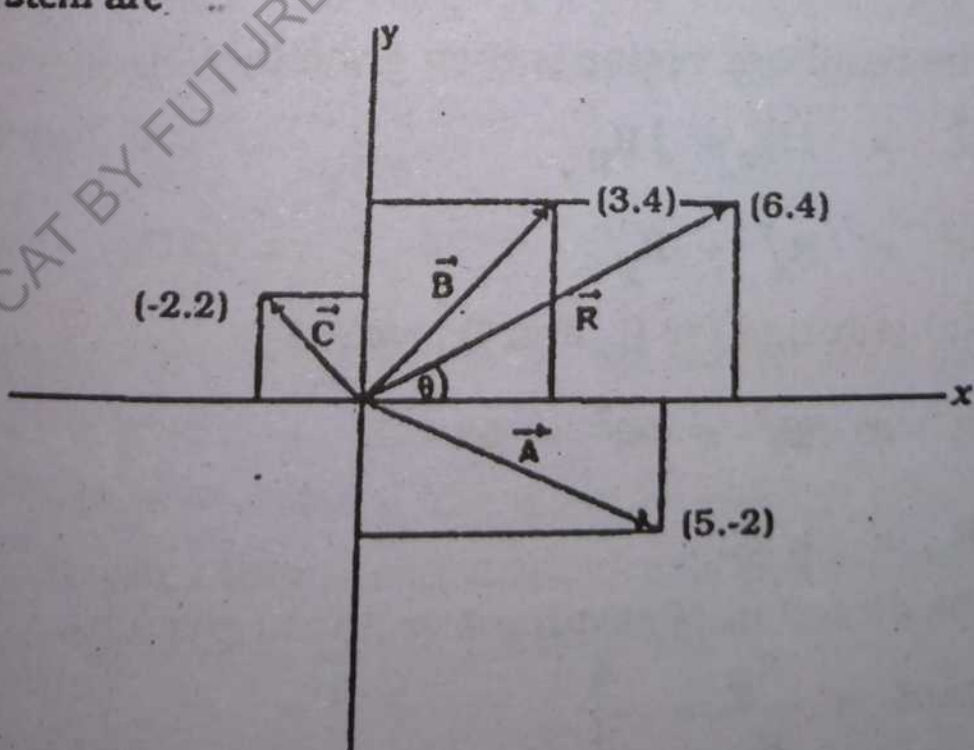


Fig. 2.18

$$\vec{A} = 5i - 2j$$

$$\vec{B} = 3i + 4j$$

$$\vec{C} = -2i + 2j$$

and the components are given by **arbitrary units. Find the resultant vector $\vec{R}$ which represents the sum of these vectors.**

Solution

The vectors $\vec{A}$, $\vec{B}$ **and** $\vec{C}$ can be expressed in terms of their components and unit vectors as

$$\vec{A} = iA_x + jA_y$$

$$\vec{B} = iB_x + jB_y$$

$$\vec{C} = iC_x + jC_y$$

then

$$R_x = A_x + B_x + C_x = 5 + 3 + (-2) = 6$$

$$R_y = A_y + B_y + C_y = -2 + 4 + 2 = 4$$

the resultant vector is then given by

$$\vec{R} = iR_x + jR_y$$

$$R^2 = R_x^2 + R_y^2$$

Substituting for R_x and R_y we get

$$R^2 = (6)^2 + (4)^2 = 52$$

$$R = \sqrt{52}$$

the direction of resultant vector is given by

$$\tan\theta = \frac{R_y}{R_x} = \frac{4}{6}$$

$$\theta = \tan^{-1}\frac{4}{6} = 33.69^\circ$$

The vectors $\vec{A}, \vec{B}, \vec{C}$, and $\vec{R}$ are drawn in Fig. 2.18 The angle θ gives the direction of the resultant vector w.r.t +ve x-axis measured counter clock wise from the +ve x-axis.

2.13 THE DOT PRODUCT

We have studied earlier the multiplication of a vector by a number. We now turn to multiplication of a vector by a vector. Like scalars, vectors of different kinds can be multiplied by one another to generate quantities of new physical dimension as explained below:

(A) SCALAR PRODUCT OF TWO VECTORS

The operation of scalar product of two vectors involves the multiplication of two given vectors in such a way that the product is a scalar.

Consider two vector $\vec{A}$ and $\vec{B}$ having magnitude A and B respectively and having angle θ between them as shown in Fig.2.19 (a). The scalar product of two vectors $\acute{A}$ and $\acute{B}$ is defined as "the product of magnitudes of the vectors and the cosine of the angle between them". Thus

$$\vec{A}.\vec{B} = AB\cos\theta\,;\quad 0 \leqq \theta \leqq \pi \qquad 2.29$$

the angle θ between $\vec{A}$ and $\vec{B}$ is the smaller angle between the positive direction of $\vec{A}$ and $\vec{B}$, i.e $\theta \leqq 2\pi - \theta$, which is inequality between two possible choices. The quantity (ABcosθ) is a scalar quantity, hence the name "scalar product". The quantity ABcosθ is also called dot product of the two vectors $\vec{A}$ and $\vec{B}$.

In Particular

(i) If $\vec{A}$ is parallel to $\vec{B}$, i.e $\theta = 0^\circ$ then

$$\vec{A}\cdot\vec{B} = AB \qquad 2.30$$

(ii) If $\vec{A} = \vec{B}$ i.e $\vec{A}$ is parallel and equal to $\vec{B}$ then

$$\vec{A}\cdot\vec{B} = AA = A^2 \quad \because \quad \theta = 0^\circ \qquad 2.31$$

(iii) If $\vec{A}$ is perpendicular to $\vec{B}$, i.e $\theta = 90^\circ$, or one of the two vectors is a null vector then

$$\vec{A}\cdot\vec{B} = 0$$

(iv) The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are perpendicular to each other therefore,

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \qquad 2.33$$

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \qquad 2.34$$

2.14 COMMUTATIVE LAW FOR DOT PRODUCT

It follows from the knowledge of projection of one vector onto the direction of another, that the scalar product of vector $\vec{A}$ and vector $\vec{B}$ is equal to the mangnitude, A, of vector $\vec{A}$ times the projection of vector $\vec{B}$ onto the direction of $\vec{A}$ as shown in Fig. 2.19(b) and vice versa as shown in Fig. 2.19(c), i.e.,

(i) From Fig: 2.19 (b)

$$\vec{A}\cdot\vec{B} = AB_A = AB\cos\theta \qquad 2.35$$

Where B_A represents the projection of vector $\vec{B}$ onto the direction of vector $\vec{A}$

(ii) From Fig: 2.19 (c)

$$\vec{B}\cdot\vec{A} = BA_B = BA\cos\theta = AB\cos\theta \qquad 2.36$$

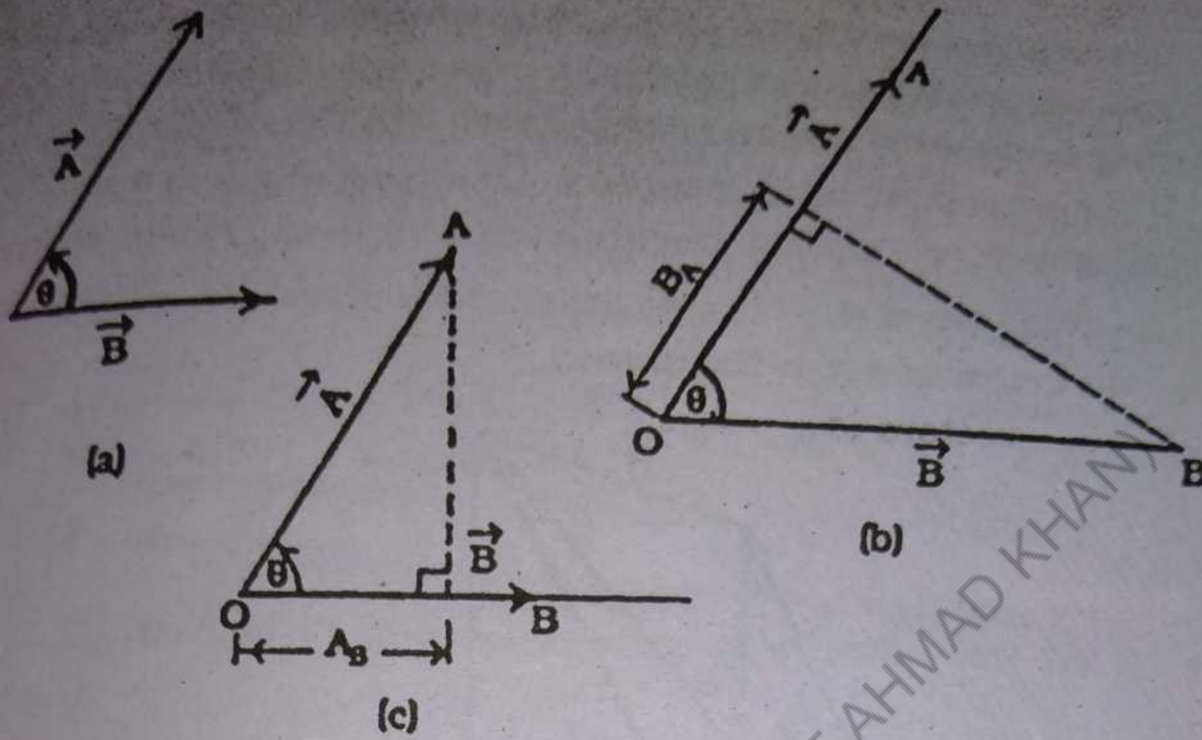


Fig. 2.19 (a) Two vectors $\vec{A}$ and $\vec{B}$ and having angle θ between them
(b) Projection of vector $\vec{B}$ onto the direction of vector $\vec{A}$
(c) Projection of vector $\vec{A}$ onto the direction of vector $\vec{B}$

where A_B represents the projection of vector $\vec{A}$ onto the direction of vector $\vec{B}$ comparing eq. 2.35 and eq. 2.36, we get

$$AB_A = BA_B$$

therefore

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

thus the scalar product of two vectors does not change with the change in the order of the vectors to be multiplied. Hence scalar product of two vectors obeys commutative law for dot product.

2.15 DISTRIBUTIVE LAW FOR DOT PRODUCT

To demonstrate the distributive law for dot product, we consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, and we use geometrical interpreta-

tion of scalar product by drawing projection as shown in Fig 2.20. First we obtain the sum of vectors $\vec{B}$ and $\vec{C}$ by Head-to-tail rule. Then we draw the projection OC_A and OR_A from the terminal points of the vector $\vec{C}$ and the vector $(\vec{B} + \vec{C})$ respectively onto the direction of vector $\vec{A}$. The dot product $\vec{A} . (\vec{B} + \vec{C})$ is equal to the projection of the vector $(\vec{B} + \vec{C})$ onto the direction of vector $\vec{A}$ multiplied by the magnitude, A, of the vector $\vec{A}$.

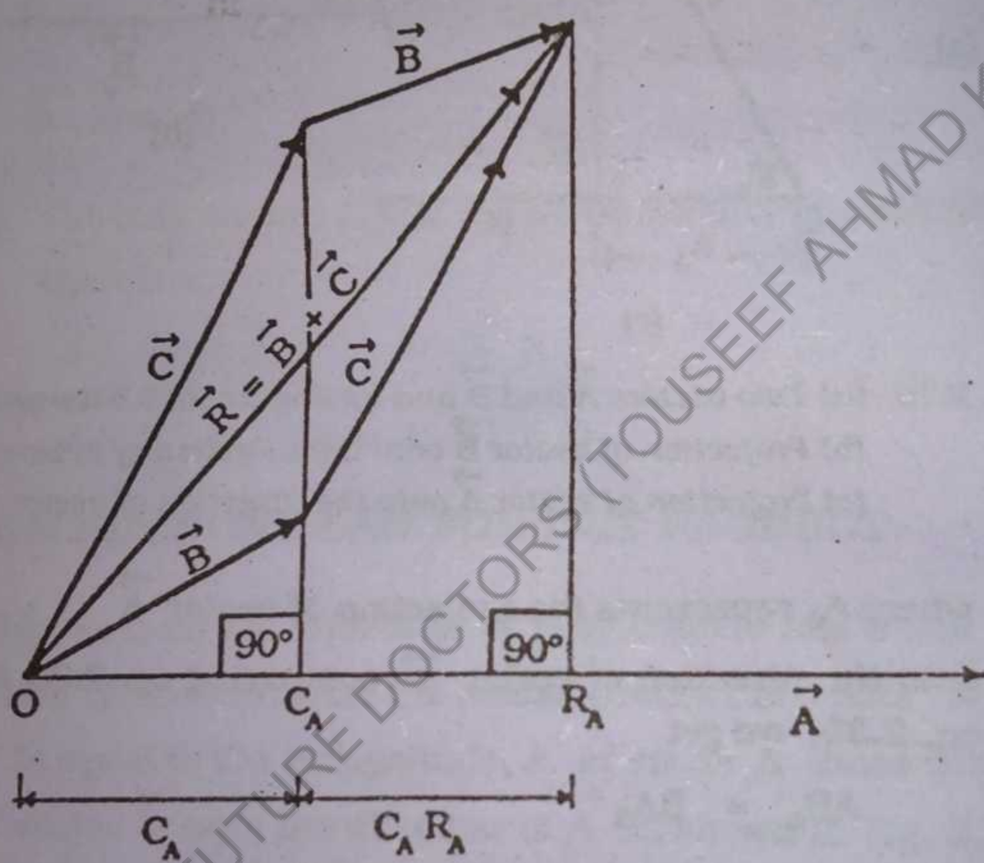


Fig. 2:20

From diagram

$$\vec{A} . (\vec{B} + \vec{C}) = A\left[OR_A\right] \quad 2.38(a)$$

$$= A\left[C_A R_A + OC_A\right] \quad 2.38(b)$$

$$= A\left[C_A R_A\right] + A\left[OC_A\right] \quad 2.38(c)$$

(i) where $C_A R_A$ - represents the projection of vector $\vec{B}$ onto the direction of $\vec{A}$

(ii) OC_A - represents the projection of vector $\vec{C}$ onto the direction of vector $\vec{A}$

therefore

$$A\left[C_A R_A\right] = \vec{A}\,.\,\vec{B} \qquad 2.38(d)$$

$$A\left[OC_A\right] = \vec{A}\,.\,\vec{C} \qquad 2.38(e)$$

substituting for $A\left[C_A R_A\right]$ & $A\left[OC_A\right]$ in Eq.No. 2.38 (c), we get

$$\vec{A}\,.\,(\vec{B}+\vec{C}) = \vec{A}\,.\,\vec{B} + \vec{A}\,.\,\vec{C} \qquad 2.38(f)$$

Eq. No.2.38 (f) demonstrates the distributive law for dot product.

Example 2.7

Evaluate the scalar product of the following:

(i) $i.i$ (ii) $i.k$ (iii) $k.(i+j)$

(iv) $(2i - j + 3k).(3i + 2j - k)$

(v) $(i - 2k).(j + 3k)$

where i, j and k represent unit vectors along x,y and z axes of three dimensional rectangular coordinate system.

Solution:

(i) $i.i = |i|\,|i|\,\text{Cos}0^\circ = 1$

(ii) $i.k = |i|\,|k|\,\text{Cos}90^\circ = 0$

(iii) $k.(i+j) = k.i + k.j$

$= |k|\,|i|\cos 90^\circ + |k|\,|j|\cos 90^\circ$

$k.(i+j) = 0$

(iv) $(2i - j + 3k).(3i + 2j - k)$

$= 6i.i + 4i.j - 2i.k - 3j.i - 2j.j + j.k +$

$9k.i + 6k.i - 3k.k$

$= 6 + 0 - 0 - 0 - 2 + 0 + 0 + 0 - 3 = 1$

(v) $(i - 2k).(j + 3k) = i.j + 3i.k - 2k.j - 6k.k$

$= 0 + 0 - 0 - 6 = -6$

Example 2.8

Find (i) the projection of $\vec{A} = 2i - 3j + 6k$ onto the direction of vector $\vec{B} = i + 2j + 2k$, (ii) determine the angle between the vectors $\vec{A}$ and $\vec{B}$.

Solution

Let a unit vector in the direction of vector $\vec{B}$ be $\hat{b}$, then by definition of a unit vector Eq.2.6

$$\hat{b} = \frac{\vec{B}}{B} = \frac{(i + 2j + 2k)}{\sqrt{(1)^2 + (2)^2 + (2)^2}} = \frac{i + 2j + 2k}{3}$$

$$\hat{b} = \frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}$$

projection of vector $\vec{A}$ onto the direction of vector $\hat{b}$ is

$$\vec{A}.\hat{b} = (2i - 3j + 6k).\left(\frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}\right)$$

$$= \frac{2}{3} - \frac{6}{3} + \frac{12}{3} = \frac{8}{3}$$

$\vec{A}.\hat{b} = 8/3$

also

$\vec{A}.\hat{b} = |\hat{b}|\,|\vec{A}|\cos\theta$

$$\cos\theta = \frac{\vec{A}.\hat{b}}{\hat{b}\,|\vec{A}|} = \frac{8/3}{|\vec{A}|}$$

$$A = |\vec{A}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = 7$$

$$\cos\theta = \frac{8/3}{7} = 8/21$$

$$\theta = 67.6^\circ$$

Example 2.9

Find the work done in moving an object along a straight line from (3,2,-1) to (2,-1,4) in a force field which is given by $\vec{F} = 4i - 3j + 2k$ and also find the angle between force and displacement.

Solution

Let W represent work which is given by.

$$W = \vec{F} \cdot \vec{d}$$

where $\vec{F}$ is applied force

$\vec{d}$ is displacement which is given by

$$\vec{d} = (x_2 - x_1)\,i + (y_2 - y_1)\,j + (z_2 - z_1)\,k$$

$$= (2 - (+3))i + (-1-(+2)\,j + (4 - (-1))\,k$$

$$= (-i) + (-3j) + (5k)$$

$$\vec{d} = (-i) + (-3j) + (5k)$$

$$W = \vec{F} \cdot \vec{d} = (4i - 3j + 2k).(-i - 3j + 5k)$$

$$= -4i.i + 9j.j + 10k.k$$

$$W = -4 + 9 + 10 = 15$$

By definition

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

Where θ is the angle between $\vec{F}$ and $\vec{d}$

$$W = Fd\cos\theta$$

to find out the angle θ we need to know the magnitudes of the vectors $\vec{F}$ and $\vec{d}$

$$F = \sqrt{(4)^2 + (-3)^2 + (2)^2} = \sqrt{29}$$

$$d = \sqrt{(-1)^2 + (-3)^2 + (5)^2} = \sqrt{35}$$

substituting for W.F and d. we get

$$\cos\theta = 15/(\sqrt{29} \times \sqrt{35})$$

$$= 0.47$$

$$\theta = 61.91°$$

Example 2.10

Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}| = 4$, $|\vec{B}| = 6$ and $\vec{A}\cdot\vec{B} = 13.5$. Find the magnitude of vector $|\vec{A} - \vec{B}|$ and the angle between $\vec{A}$ and $\vec{B}$.

Solution

$$\because |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

$$|\vec{A} - \vec{B}| = \sqrt{(\vec{A} - \vec{B})\cdot(\vec{A} - \vec{B})}$$

$$= \sqrt{\vec{A}\cdot\vec{A} - 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}}$$

The expression $\vec{A}\cdot\vec{A} = |\vec{A}|^2 = A^2$.

Similarly $\vec{B}\cdot\vec{B} = |\vec{B}|^2$, then

$$|\vec{A} - \vec{B}| = \sqrt{|\vec{A}|^2 \quad 2\vec{A}\cdot\vec{B} + |\vec{B}|^2}$$

$$= \sqrt{(4)^2 - 2 \times 13.5 + (6)^2} = 5$$

Also

$$\vec{A}\cdot\vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{13.5}{4\times 6} = 13.5/24$$

$$\theta = 55.77^\circ$$

2.16 THE CROSS PRODUCT

In preceding sections, we developed and discussed the concept of multiplication of two vectors in such a way that their resultant product is a scalar quantity. When dealing with quantities such as torque, angular momentum, the force on a moving charge in a magnetic field, flow of electro magnetic energy, etc, we turn to the multiplication of two given vectors in such a way that the resultant product is a vector quantity. This product is known as vector product or cross product.

Consider two vectors $\vec{A}$ and $\vec{B}$, the vector product of these two vectors is denoted by $\vec{A}\times\vec{B}$, and read as "$\vec{A}$ cross $\vec{B}$". The cross or vector product of $\vec{A}$ and $\vec{B}$, is a new vector $\vec{C} = \vec{A}\times\vec{B}$, by definition the vector $\vec{C}$ is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$. By definition (i) the magnitude, $|\vec{A}\times\vec{B}|$, of the cross product or the magnitude, $|\vec{C}|$, of the vector $\vec{C}$ is given by

$$C = |\vec{C}| = AB\sin\theta,\ 0 \leq \theta \leq \pi \qquad 2.39$$

Where $\vec{A}$ and $\vec{B}$, represent the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively. θ is smaller angle between the positive direction of $\vec{A}$ and $\vec{B}$ i.e. $\theta \leq 2\pi - \theta$.

(ii) The vector $\vec{C} = \vec{A}\times\vec{B}$, which represents the cross or vector product is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ (by definition) and points in the direction in such a way as to make $\vec{A}$, $\vec{B}$, and $\vec{C}$, in that order, a right handed system as shown in Fig.2.21 (a).

We generalize this definition and write

$$\vec{C} = \vec{A}\times\vec{B} = [AB\sin\theta]\,\hat{u} \qquad 2.40$$

Where $\hat{u}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ and in the sense determined by a right handed screw turning from $\vec{A}$ to $\vec{B}$.

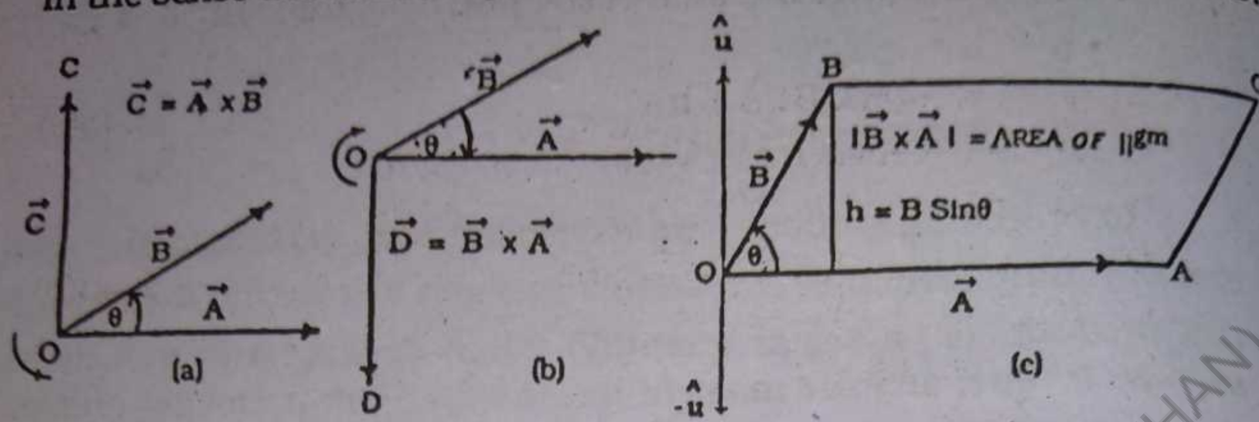


Fig. 2.21 shows vector product $\vec{A} \times \vec{B}$ In right handed coordinate system,
Fig. 2.21(a) shows the direction of the vector $\vec{C}$ is that in which a right handed screw advances when turned from $\vec{A}$ to $\vec{B}$
Fig 2.21 (b) shows the direction of the vector $\vec{D}$ changes through 180° when turned from $\vec{B}$ to $\vec{A}$.
Fig. 2.21 (c)The area of Parallelogram is given by the magnitude of the cross product

Similarly a right handed screw turning from $\vec{B}$ to $\vec{A}$ defines the unit vector $-\hat{u}$, then

$$\vec{D} = (\vec{B} \times \vec{A}) = [B\,A\,\sin\theta]\,(-\hat{u}) \qquad 2.41(a)$$

$$= (\vec{B} \times \vec{A}) = -[B\,A\,\sin\theta]\,(\hat{u})$$

$$-\vec{D} = -(\vec{B} \times \vec{A}) = [B\,A\,\sin\theta]\,\hat{u} \qquad 2.41(b)$$

The quantities AB sin θ and BA sin θ on the R.H.S. of Eq 2.40 and 2.41 (b) being the magnitudes are equal therefore

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \text{ or } \vec{C} = -\vec{D} \qquad 2.42$$

The Eq: 2.42 signifies that the commutative law for cross product is not valid.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{or} \quad \vec{C} = -\vec{D}$$

The Eq. 2.42 signifies that vector multiplication is not commutative.

To give physical interpretation of vector product of two given vectors, we consider two vectors $\vec{A}$ and $\vec{B}$. Let these vectors repre-

sent two adjacent sides of a parallelogram OBCA as shown in Fig.2.21(c). From figure, the area of the parallelogram OBCA is given by

the area of ||gm OBCA = hA 2.43(a)

if the angle between two vectors is θ, then

$h = B \sin\theta$

the area of ||gm OBCA = A B sinθ 2.43(b)

By definition we know

(i) $C = AB\sin\theta$ 2.44

and

$$\vec{C} = \vec{A} \times \vec{B} \qquad 2.45$$

(ii) the vector product, $A \times \vec{B}$, is perpendicular to the plane containing both $\vec{A}$ and $\vec{B}$ Comparing eq. 2.43(b) and Eq 2.44, we conclude that the cross product is perpendicular to the parallelogram defined by vector $\vec{A}$ and Vector $\vec{B}$ and its magnitude is equal to the area of the parallelogram.

2.17 SOME PHYSICAL EXAMPLES OF VECTOR PRODUCT.

(i) The simplest example of a vector product is the moment M of a force about a point O, defined as

$$\vec{M} = \vec{R} \times \vec{F}$$

Where $\vec{R}$ is a vector joining the point 'O' to the initial point of $\vec{F}$

(ii) An electric charge, q, moving with velocity $\vec{V}$ in a magnetic field $\vec{B}$ experiences a force $\vec{F}$, which is given by

$$\vec{F} = q(\vec{V} \times \vec{B}).$$

2.18 PROPERTIES OF THE VECTOR PRODUCT

The following are the important properties of vector product:

(i) $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$ 2.46

(ii) $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$ 2.47

(iii) $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$ 2.48

(iv) If $\vec{A} \neq 0, \vec{B} \neq 0$ and $\vec{A} \times \vec{B} = 0$, then

$\vec{A}$ and $\vec{B}$ are parallel.

(v) $i \times i = 0$ 2.49

$j \times j = 0$ 2.50

$k \times k = 0$ 2.51

(vi) $i \times j = k$ 2.52

$j \times k = i$ 2.53

$k \times i = j$ 2.54

(vii) $i \times j = -j \times i = k$ 2.55(a)

$j \times k = -k \times j = i$ 2.56(b)

$k \times i = -i \times k = j$ 2.57(c)

(viii) If $\vec{A} = A_1 i + A_2 j + A_3 k$

$\vec{B} = B_1 i + B_2 j + B_3 k$

the cross product or vector product will be written as

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Example 2.11

$\vec{R}_1$ and $\vec{R}_2$ are two position vectors making angle θ_1 and θ_2 with positive x-axis respectively. Find their vector product when

$R_1 = 4\text{cm}, R_2 = 3\text{ cm}$

$\theta_1 = 30°, \theta_2 = 90°$

Solution

The angle between two vectors is $\theta = \theta_2 - \theta_1 = 60°$ the magnitude of the cross product of vectors $\vec{R}_1$ and $\vec{R}_2$ is

$$C = R_1 \; R_2 \; \sin\theta$$

$$= 4 \times 3 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

As the vectors $\vec{R}_1$ and $\vec{R}_2$ lie in x-y plane, therefore the vector representing cross product lie parallel to z-direction.

Example 2.12

Two sides of a triangle are formed by the vector $\vec{A} = 3i + 6j - 2k$ and vector $\vec{B} = 4i - j + 3k$. Determine the area of the triangle.

Solution

The area of triangle in terms of vector product is given by $\frac{1}{2}\,|(\vec{A} \times \vec{B})|$.

$$\vec{A} \times \vec{B} = (3i + 6j - 2k) \times (4i - j + 3k)$$

$$= 3i \times (4i - j + 3k) + 6j \times (4i - j + 3k) - 2k \times (4i - j + 3k)$$

$$= 12i \times i - 3i \times j + 9i \times k + 24j \times i - 6j \times j + 18j \times k - 8k \times i + 2k \times j - 6k \times k$$

$$\vec{A} \times \vec{B} = 16i - 17j + 27k$$

$$C = |\vec{A} \times \vec{B}| = \sqrt{(16)^2 + (-17)^2 + (-27)^2} = \sqrt{1274}$$

$$\frac{1}{2} C = \frac{1}{2}\sqrt{1274}$$

which is the required area of triangle Alternatively.

$$\vec{A} \times \vec{B} = (3i + 6j - 2k) \times (4i - j + 3k)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 3 & 6 & -2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 6 & -2 \\ -1 & 3 \end{vmatrix} + j \begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} + k \begin{vmatrix} 3 & 6 \\ 4 & -1 \end{vmatrix}$$

$$= i(18-2) + j(-8-9) + k(-3-24)$$

$$= 16i - 17j - 27k$$

$$C = \sqrt{(16)^2 + (-17)^2 + (-27)^2} = \sqrt{1274}$$

$$\frac{1}{2} C = \frac{1}{2} \sqrt{1274}$$

Example 2.13

Determine a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, if $\vec{A} = 2i - 3j - k$, $\vec{B} = i + 4j - 2k$

Solution

By definition $\vec{A} \times \vec{B}$ is vector which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and

$$\vec{C} = \vec{A} \times \vec{B} = (2i - 3j - k) \times (i + 4j - 2k)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= i \begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} + j \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

$$= i(6+4) + j(-1+4) + k(8+3)$$

$$= 10i + 3j + 11k$$

A u

$\hat{C} =$

$\hat{C} =$

$\hat{C} =$

PR

1.

vectors.

(1)
(3)
(5)
(7)
(9)
(11)
(13)
(15)
(17)
(19)
Ans:

2. Fin

A unit vector parallel to $\vec{A} \times \vec{B}$ is given by Eq 2.6

$$\hat{C} = \frac{\vec{C}}{C} = \frac{10i + 3j + 11k}{\sqrt{(10)^2 + (3)^2 + (11)^2}}$$

$$\hat{C} = \frac{10i + 3j + 11k}{\sqrt{230}}$$

$$\hat{C} = \frac{\vec{C}}{C} = \frac{10i}{\sqrt{230}} + \frac{3j}{\sqrt{230}} + \frac{11k}{\sqrt{230}}$$

PROBLEMS:-

1. State which of the following are scalars and which are vectors.

(1) Weight
(2) Calorie
(3) Specific heat
(4) Momentum
(5) Density
(6) Energy
(7) Volume
(8) Distance
(9) Speed
(10) Magnetic field intensity
(11) Entropy
(12) Work
(13) Centrifugal force
(14) temperature
(15) gravitational potential
(16) Charge
(17) Shearing stress
(18) frequency
(19) Kinetic energy
(20) Electric field intensity.

Ans:

(1) vector	(2) scalar	(3) scalar
(4) vector	(5) scalar	(6) scalar
(7) scalar	(8) scalar	(9) scalar
(10) vector	(11) scalar	(12) scalar
(13) vector	(14) scalar	(15) scalar
(16) scalar	(17) vector	(18) scalar
(19) scalar	(20) vector	

2. Find the resultant of the following displacement:

$\vec{A}$ = 20 Km 30° south of east:

$\vec{B}$ = 50km due west

$\vec{C}$ = 40 km north east;

$\vec{D}$ = 30 Km 60° south of west.

Ans: Magnitude 20.9 km. direction 21.65° south of west

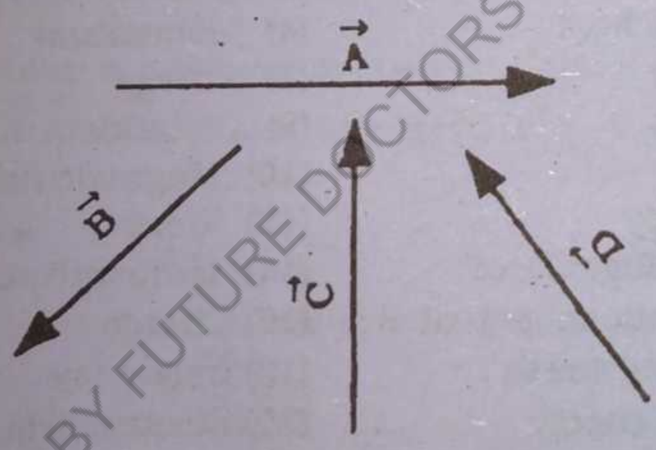
3. An aeroplane flies 400 km due west from city A to city B, then 300 km north east to city C, and finally 100 km north to cityD. How far is it from city A to D? In what direction must the aeroplane had to return directly to city A from city D?

Ans: 364 km at 31° east of south.

4. Show graphically that $-(\vec{A} - \vec{B}) = -\vec{A} + \vec{B}$

5. Given vector $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ as shown in figure below. Construct (a) $4\vec{A} - 3\vec{B} - (2\vec{C} + 2\vec{D})$

(b) $(1/2)(\vec{C}) + (1/3)(\vec{A} + \vec{B} + 2\vec{D})$



6. The following forces act on a particle P:

$\vec{F}_1 = 2i + 3j - 5k$, $\vec{F}_2 = -5i + j + 3k$,

$\vec{F}_3 = i - 2j + 4k$, $\vec{F}_4 = 4i - 3j - 2k$,

measured in newtons Find (a) the resultant of the forces

(b) the magnitude of the resultant force

Ans. (a) $2i - j$ (b) $\sqrt{5}$

7. If $A = 3\hat{i} - \hat{j} - 4\hat{k}$, $B = -2\hat{i} + 4\hat{j} - 3\hat{k}$
$C = \hat{i} + 2\hat{j} - \hat{k}$, find

(a) $2\vec{A} - \vec{B} + 3\vec{C}$, (b) $|\vec{A} + \vec{B} + \vec{C}|$,

(c) $|3\vec{A} - 2\vec{B} + 4\vec{C}|$

(d) a unit vector parallel to $3\vec{A} - 2\vec{B} + 4\vec{C}$.

Ans. (a) $9i - 4j - 6k$ (b) $\sqrt{93}$ (c) $\sqrt{398}$ (d) $(3\vec{A} - 2\vec{B} + 4\vec{C})/\sqrt{398}$

8. Two tugboats are towing a ship. Each exerts a force of 6000N, and the angle between the two ropes is 60°. Calculate the resultant force on the ship.

Ans. 10392 N

9. The position vectors of points P and Q are given by $\vec{r}_1 = 2i + 3j - k$, $\vec{r}_2 = 4i - 3j + 2k$. Determine $\overrightarrow{PQ}$ in terms of rectangular unit vector i, j and k and find its magnitude.

Ans. $2i - 6j + 3k$, 7

10. Prove that the vectors $\vec{A} = 3i + j - 2k$,

$\vec{B} = -i + 3j + 4k$, $\vec{C} = 4i - 2j - 6k$,

can form the sides of a triangle. Find the length of the medians of the triangle.

Ans: $\sqrt{6}$, $\frac{1}{2}\sqrt{114}$, $\frac{1}{2}\sqrt{150}$

11. Find the rectangular components of a vector $\vec{A}$, 15 unit long when it form an angle with respect to +ve x-axis of (i) 50°, (ii) 130° (iii) 230°, (iv) 310°

Ans. (i) A_x = 9.6 unit, A_y = 11.5 unit

(ii) A_x = -9.6 unit, A_y = 11.5 unit

(iii) $A_x = -9.6$ unit. $A_y = -11.5$ unit.

(iv) $A_x = 9.6$ unit $A_y = -11.5$ unit.

12. Two vectors 10cm and 8cm long form an angle of (a) 60°, (b) 90° and (c) 120°. Find the magnitude of difference and the angle with respect to the larger vector.

Ans. 9.2 cm, 49° (b) 12.8 cm, 38° 41′ (c) 15.6 cm, 26°.22′

13. The angle between the vector $\vec{A}$ and $\vec{B}$ is 60°. Given that $|\vec{A}| = |\vec{B}| = 1$, calculate (a) $|\vec{B}-\vec{A}|$; (b) $|\vec{B}+\vec{A}|$

Ans (a) 1, (b) $\sqrt{3}$

14. A car weighing 10,000 N on a hill which makes an angle of 20° with the horizontal. Find the components of car's weight parallel and perpendicular to the road.

Ans. $F_{\parallel} = 3420$ N, $F_{\perp} = 9400$N

15. Find the angle between $A = 2\hat{i} + 2\hat{j} - \hat{k}$ and $B = 6\hat{i} - 3\hat{j} + 2\hat{k}$.

Ans. $\theta = 79°$

16. Find the projection of the vector $A = \hat{i} - 2\hat{j} + \hat{k}$ onto the direction of vector $B = 4\hat{i} - 4\hat{j} + 7\hat{k}$.

Ans. 19/09

17. Find the angles α, β, γ which the vector $A = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the positive x,y,z axis respectively.

Ans. $\alpha = 64.6°$, $\beta = 149°$, $\gamma = 73.4°$

18. Find the work done in moving an object along a vector $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$.

Ans.9

19. Find the work done by a force of 30.000N in moving an object through a distance of 45m when: (a) the force is in the direction of motion; and (b) the force makes an angle of 40° to the direction of motion. Find the rate at which the force is working at a time when the velocity is 2m/sec.

Ans. (a) 1.35×10^6 J, 6×10^4 w (b) 1.03×10^6 J, 4.60×10^4 w.

20. Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}| = 3$, $|\vec{B}| = 4$, and $\vec{A}.\vec{B}. = -5$, find

(a) the angle between $\vec{A}$ and $\vec{B}$

(b) the length $|\vec{A} + \vec{B}|$ and $|\vec{A} - \vec{B}|$

(c) the angle between $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$

Ans. (a) 114.6° (b) $\sqrt{15}$, $\sqrt{35}$ (c) 107°.8

21. If $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$, $\vec{B} = \hat{i} + 4\hat{j} - 2\hat{k}$.

Find (a) $\vec{A} \times \vec{B}$. (b) $\vec{B} \times \vec{A}$. (c) $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B})$

Ans. (a) $10\hat{i} + 3\hat{j} + 11\hat{k}$ (b) $-10\hat{i} - 3\hat{j} - 11\hat{k}$

(c) $-20\hat{i} - 6\hat{j} - 22\hat{k}$

22. Determine the unit vector perpendicular to the plane of $\vec{A} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} - \hat{k}$

Ans. $+(\frac{3}{7}\hat{i} \quad \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k})$

23. Using the definition of vector product, prove the law of sines for plane triangles of sides a, b and c.

Ans. sin A/a = sin B/b = sin C/c

24. If $\vec{r}_1$ and $\vec{r}_2$ are the position vectors (both lie in xy plane) making angle θ_1 and θ_2 with the positive x - axis measured counter clockwise, find their vector product when

(i) $|\vec{r}_1| = 4$ cm $\qquad \theta_1 = 30^\circ$

$|\vec{r}_2| = 3$ cm $\qquad \theta_2 = 90^\circ$

(ii)	$\lvert\vec{r}_1\rvert = 6$ cm		$\theta_1 = 220^\circ$
	$\lvert\vec{r}_2\rvert = 3$ cm		$\theta_2 = 40^\circ$
(iii)	$\lvert\vec{r}_1\rvert = 10$ cm		$\theta_1 = 20^\circ$
	$\lvert\vec{r}_2\rvert = 9$ cm		$\theta_2 = 110^\circ$

Ans. (i) $6\sqrt{3}$ cm^2 (ii) 0. (iii) 90 cm^2

Motion

3.1 DISPLACEMENT

The change of position of a body in a particular direction is called it displacement. By definition it is a vector quantity. If a body moves from a position A to another position B as shown in Fig.3.1(a) we can represent its displacement by drawing a line from A to B. The direction of displacement can be shown by putting an arrow head at B, which indicates the direction of displacement

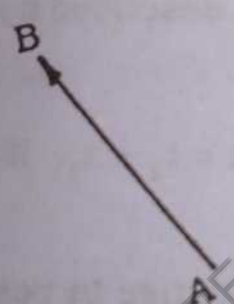


Fig. 3.1 (a)

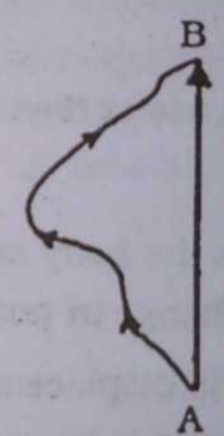


Fig: 3.1 (b)

from A to B. The actual path of a body may not be a straight line from A to B, it may be a curved path as shown in fig.3.1 (b). The arrow represents the direction of motion of the body.

3.2 VELOCITY

The velocity of a body is defined as the change of its displacement with respect to time. Alternatively it is also defined as the rate of change of its position in a particular direction.

Consider a body in motion. The path of its motion is represented by line AC as shown in Fig 3.2

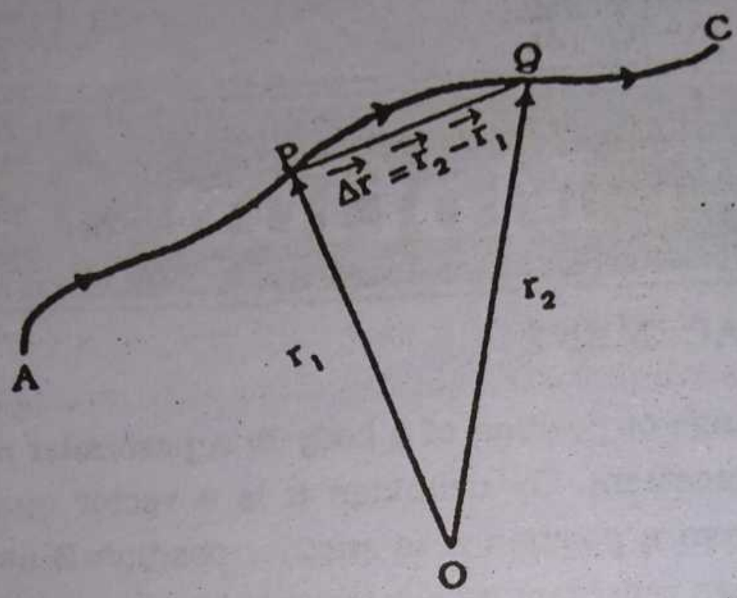


Fig. 3.2

At time t_1, let the body be at point P in Fig 3.2. Its position at this instant with respect to origin O, is represented by vector $\overrightarrow{OP} = \vec{r}_1$.

At a later time t_2, let the body be at point Q, described by vector $\vec{r_2}$

As the body moves from P to Q in time $\Delta t = t_2 - t_1$, it undergoes a change in position, $\overrightarrow{\Delta r} = (\vec{r}_2 - \vec{r}_1)$.

The displacement vector describing the change in position of the body as it moves from P to Q is $\overrightarrow{\Delta r}$ which is equal to $(\vec{r}_2 - \vec{r}_1)$ i.e $\overrightarrow{\Delta r} = (\vec{r}_2 - \vec{r}_1)$., and the time for the motion between these two points is Δt, which is equal to $(t_2 - t_1)$ i.e $\Delta t = (t_2 - t_1)$. The average velocity of the body during this interval is defined by

$$\vec{v}_{av} = \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{\text{displacement}}{\text{time}} \qquad 3.1$$

The rate of change of position of a body in the direction of displacement is called velocity.

If the time is very small such that . $\Delta t \to o$.

$$\vec{v}_{ins} = \lim_{\Delta t \to o.} \frac{\overrightarrow{\Delta r}}{\Delta t} \qquad 3.2$$

Thi. velocity is called instantaneous velocity.

Whenever the average and instantaneous velocities are equal the body is said to have a uniform velocity.

That is. a body is said to have uniform velocity if it travels equal distances in equal intervals of time in a given direction however the small interval may be. The S.I unit of velocity is metre per second (m/s).

3.3 VELOCITY FROM DISTANCE - TIME GRAPH:-

The velocity of a body can be determined by distance time graph also.

When a body moves with uniform velocity it will travel equal distances in equal intervals of time. A graph of distance against time will be a straight line as shown in fig. 3.3 (a). If we take any point A, on the graph and draw a perpendicular AB on the time axis, it is

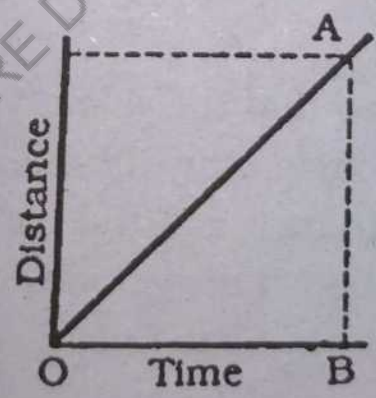


Fig: 3.3 (a) Uniform Velocity

clear that AB represents the distance travelled in the time interval represented by OB, hence

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}} = \frac{AB}{OB} \qquad 3.3$$

The ratio $\frac{AB}{OB}$ is called the velocity of the body.

Fig. 3.3 (b) represents a graph of distance travelled in a given time by a body moving with variable velocity. If we want to find the velocity of the body at any point, say A on the curve then we draw a tangent EG to the curve at point A and obtain a right angled triangle E F G and measuring its slope.

$$\text{Velocity at } A = \frac{GF}{EF} \quad 3.4$$

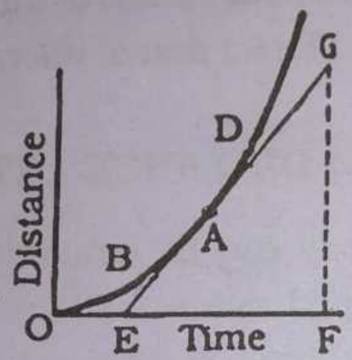


Fig: 3.3 (b) Non-Uniform Velocity

3.4 ACCELERATION :-

We have just defined that the velocity of a body is the distance covered by it in a particular direction in unit time. It can be changed by a change in its magnitude or its direction or both. When ever there is a change in the velocity of a body, the body is said to possess acceleration. By definition acceleration is a vector quantity.

Suppose that at any instant t_1, the body is at point A and is moving with velocity $\vec{V}_1$, At a later time t_2 it is at point B moving with velocity $\vec{V}_2$.

The average acceleration $\vec{a}_{av}$ during the motion from A to B is defined to be the change of velocity divided by the time interval of

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\overrightarrow{\Delta v}}{\Delta t} \qquad 3.5$$

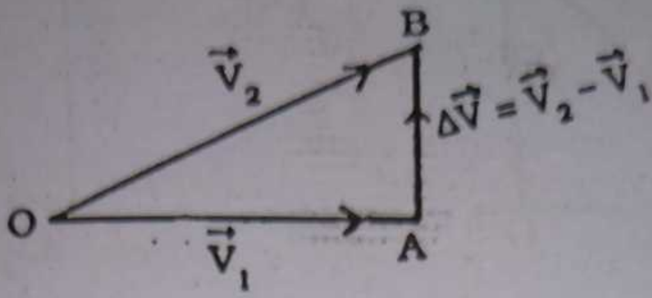


Fig. 3.4

Acceleration being a vector quantity has the same direction as that of $\overrightarrow{\Delta v}$

In the limits of a very small Δt the average acceleration will approach the value of instantaneous acceleration. Thus the instantaneous acceleration, $\vec{a}_{ins}$ is defined as

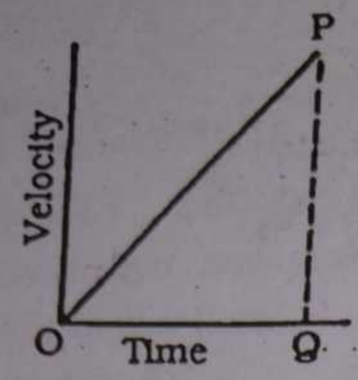
$$\vec{a}_{ins} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} \qquad 3.6$$

If the velocity of a body is decreasing the acceleration is negative. the negative acceleration is also known as retardation or deceleration.

The S.I unit of acceleration is metre per second per second and written as m/s^2

3.5 ACCELERATION FROM VELOCITY - TIME GRAPH:

Figure 3.5 shows that velocity - time graph for a body moving. in a straight line with (a) uniform acceleration and (b) non-uniform acceleration. A displacement-time graph was used to determine the value of velocity. similarly we can use velocity-time graph for the value of acceleration.



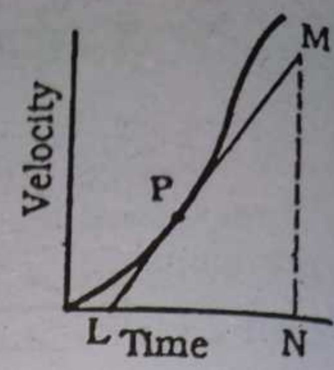


Fig. 3.5 (a) uniform acceleration *(b) Non Uniform acceleration*

From Fig 3.5(a) and 3.5(b)

$$\text{Acceleration} = \frac{PQ}{OQ} \qquad 3.7$$

$$\text{Acceleration at P} = \frac{MN}{LN} \qquad 3.8$$

3.6 EQUATIONS OF UNIFORMLY ACCELERATED RECTILINEAR MOTION

We have studied that if a body is moving with constant acceleration a, its initial velocity is V_i and after time 't' its final velocity is V_f. Then the motion of the body is governed by the following equations.

$$V_f = V_i + at \longrightarrow (1)$$

$$S = V_i t + \frac{1}{2} at^2 \longrightarrow (2)$$

$$V_f^2 = V_i^2 + 2aS \longrightarrow (3)$$

Example 3.1

As the traffic light turns green, a car starts from rest with a constant acceleration of 4 m/s^2. At the same time, a motorcyclist travelling with a constant speed of 36 Km/h, overtakes and passes the car. Find (a) How far beyond the starting point will the car overtakes the motorcyclist (b) what will be speed of the car at the time when it overtakes the motorcycle.

Solution:-

(a) Suppose the car overtakes the motorcyclist at a distance "S" from the starting point of the car.

Now initial velocity of the car = V_i = $0\ ms^{-1}$

Acceleration of the car = a = $4\ ms^{-2}$

Distance covered = S = ?

Let t be the time during which this distance "S" is covered by the car.

Applying the eq: 2

$$S = V_i t + \frac{1}{2} at^2$$

$$= 0 \times t + \frac{1}{2} \times 4 \times t^2$$

$$S = 2t^2 \longrightarrow (1)$$

Similarly for the motorcyclist

speed of the motorcyclist = υ = 36 $km/_h$ = 10 m/s

Time = t seconds.

Applying eq.

$$S = V \times t$$

We get $S = 10t \longrightarrow (2)$

Now equating Eq. (1) and Eq (2) we get

$$2t^2 = 10t$$

$$\therefore t = 5\ sec$$

Putting this value in Eq (2), we get

$$S = V \times t$$

$$= 10 \times 5 = 50\ m.$$

Hence the car will overtake the motorcyclist at a distance of 50 metres.

(b) Let V_f be the velocity with which the car overtakes the motorcyclist.

$$V_i = 0$$

$$V_f = ?$$

$$t = 5\ s.$$

$$a = 4\ m/s^2$$

$$V_f = V_i + at$$

$$= 0 + 5 \times 4$$

$$= 20\ m/s$$

The speed of the car at the time of overtaking is 20 m/s

Example 3.2

A car starts from rest and moves with a constant acceleration. During the 5th second of its motion, it covers a distance of 36 metres. Calculate (a) the acceleration of the car (b) the total distance covered by the car during this time.

Solution:-

Suppose the total distance covered by the car is "S". Let "a" be the acceleration.

Now

Distance covered by the car in 5 seconds = S_5

Initial velocity of the car = V_i = 0

Acceleration = a = ?

Time taken = 5 seconds

Using Eq.

$$S = V_i t + \frac{1}{2} at^2$$

we get

$$S_5 = 0 \times 5 + \frac{1}{2} \times a \times (5)^2$$

$$= \frac{25}{2} a$$

$$= 12.5\ a \longrightarrow \quad (1)$$

Similarly distance covered by the car in 4 seconds = S_4

Initial Velocity of the car = V_i = 0

Acceleration = a = ?

Time taken = 4 seconds.

Using the Eq.

$$S = V_i t + \frac{1}{2} at^2$$

we get

$$S_4 = \times 4 + \frac{1}{2} \times a \times (4)^2$$

$$S_4 = 8a \longrightarrow \quad (2)$$

Now the distance covered by the car in 5th second

$$= S_5 - S_4 = 36\text{ m}$$

Subtracting Eq. (2) from Eq. (1) and putting these values we get

$$S_5 \quad S_4 = 12.5a - 8a$$

$$36 = 4.5a$$

$$a = \frac{36}{4.5} = 8\text{ m/s}^2$$

(b) The total distance covered by the car in 5 seconds = S = ?

Initial velocity of the car = V_i = 0

Acceleration = a = 8 m/s^2

Time = t = 5 seconds

Applying the equation

$$S = V_i t + \frac{1}{2} at^2$$

we get

$$S_5 = 0 \times 5 + \frac{1}{2} \times 8 \times (5)^2$$
$$= 0 + 4 \times 25$$
$$= 100m$$

3.7 MOTION UNDER GRAVITY

The most common example of motion with nearly constant acceleration is that of a body falling towards the earth. This acceleration is due to pull of the earth (gravity). If the body moves towards earth, neglecting air resistance and small changes in the acceleration with altitude. This body is referred to as free falling body and this motion is called Free Fall.

Such type of vertical motion under the action of gravity is good example of uniformly accelerated motion. The acceleration due to gravity is usually represented by g. Replacing acceleration a by acceleration due to gravity 'g' the equations of motion becomes

$$V_f = V_i + gt$$

$$S = V_i t + \frac{1}{2} gt^2$$

$$V_f^2 = V_i^2 + 2gS$$

In S.I system the value of "g" is 9.8 m/s^2

Example 3.3

A ball is dropped from the top of a tower. If it takes 5 seconds to hit the ground, find the height of the tower and with what velocity will it strike the ground.

Initial velocity $= V_i = 0$

Acceleration $= a = + g = 9.8 \text{ ms}^{-2}$

(The acceleration is positive because the direction of initial motion is down ward)

Height of the tower = S = h

Time = t = 5 seconds

To find the height of the tower we will use the equation

$$S = V_i t + \frac{1}{2} g t^2$$

$$= 0 \times 5 + \frac{1}{2} \times +9.8 \times (5)^2$$

$$= 122.5 \text{ metres}$$

Let V_f be the velocity of the ball with which it strikes the ground so.

$V_f = ?$

$V_i = 0$

$t = 5$ seconds

$a = g = 9.8 \text{ m/s}^2$

Applying Eq.1

$$V_f = V_i + gt$$

$$= 0 + 9.8 \times 5$$

$$= 49.00 \text{ m/s}$$

Example 3.4

A ball is thrown vertically upward with a velocity of 98 m/s

(a) How high does the ball rise?

(b) How long does it take to reach its highest point?

(c) How long does the ball remain in air?

(d) With what speed does the ball return to the ground?

Case (a)

At the highest point, the velocity of the ball is zero. If "h" is the distance covered by the ball then its value can be obtained by applying the equation

$$V_f^2 = V_i^2 + 2aS$$

here $V_f = 0\ ms^{-1}$

$V_i = 93\ ms^{-1}$

$a = -g \quad -9.8\ ms^{-2}$

$S = h = ?$

Acceleration in this case is taken to be negative because the initial velocity is directed upward.

Now applying the equation we get

$$V_f^2 - V_i^2 = 2aS$$

$$(0)^2 - (98)^2 = 2(-9.8) \times h$$

$$-98 \times 98 = -19.6 \times h.$$

$$19.6\,h = 98 \times 98$$

$$h = \frac{98 \times 98}{19.6} = 490 \text{ metres}$$

Case (b) If "t" is the time taken by the ball to reach the highest point, its value is obtained by equation

$$V_f = V_i + at$$

Here initial velocity $V_i = 98\ m/s$

Final velocity $V_f = 0\ m/s$

Acceleration $a = -g = -9.8\ m/s^2$

Time $t = ?$

putting these values in Eq

$$V_f = V_i + at$$

we get

$$0 = 98 + (-9.8) \times t$$
$$9.8 \times t = 98$$
$$\therefore t = \frac{98}{9.8} = 10 \text{ seconds}$$

Case (c) The ball remains in air during the time interval in which it goes from the point of projection to the highest point and then comes back to its initial position, Let this interval be T seconds. The net displacement after T seconds is zero. The value of T can be calculated by the equation

$$S = V_i t + \frac{1}{2} a t^2$$

Here initial velocity $V_i = 98 \text{ m/s}$

Acceleration $= a = -g = -9.8 \text{ m/s}^2$

Time $= t = T = ?$

Displacement $= S = 0$

$$0 = 98T - \frac{1}{2} \times 9.8 \times T^2$$
$$0 = 98T - 4.9T^2$$
$$4.9T^2 - 98T = 0$$
$$T = 20 \text{ seconds}$$

Note:-

We have seen in case b that the time taken by the ball to reach its highest point is 10 seconds. Thus the time taken by the ball to come down from its highest point to the point of projection is 20 - 10 = 10 seconds. Thus we see that the time of upward motion is the same as that of downward motion.

Case (d):-

When the ball returns to the ground, its net displacement is zero and the velocity with which it returns to the ground can be calculated by the equation

$$V_f^2 - V_i^2 = 2aS$$

Here

Acceleration $= a = -g = -9.8\ m/s^2$

Initial velocity $= V_i = 98\ m/s$

Final velocity $= V_f = ?$

Displacement $= S = 0$

$$\therefore V_f^2 - (98)^2 = 2 \times (-9.8) \times (0)$$

$$\therefore V_f^2 = (98)^2$$

$$\therefore V_f = \pm 98\ m/s$$

The value + 98 m/s corresponds to the instant when the ball was projected up and thus -98 m/s is the velocity with which the ball returns to the ground. The negative sign tells us that the direction of motion of the ball, when it returns is opposite to that of initial velocity. So the ball returns with a speed of 98 m/s in the downward direction.

3.8 LAWS OF MOTION

Issac Newton studied motion of bodies and formulated the following laws:

(i) Newton's First Law of Motion.

"A body remains at rest or continues to move with uniform velocity unless acted upon by an unbalanced force".

The law consists of two parts: (i) the first part states that a body cannot change its state of rest or uniform motion in a straight line itself unless it is acted upon by some unbalanced force to

change its state. It can also be stated that a moving body when not acted upon by some net force would have free motion, that is, uniform motion in a straight line.

The second part of this law gives us the qualitative definition of the net force, which is stated as follows. Force is an agency which when applied to a body, changes or tends to change its state of rest or of uniform motion i.e produces acceleration in the body.

This law is also called the law of inertia because it points towards a very important property of matter which is called inertia.

Inertia is that property of matter by virtue of which if it is in state of rest or motion it tries to remain in that state.

If two bodies of different masses are moving with the same velocity under identical conditions, it will be more difficult to stop or change the motion of the body of the larger mass, because the body with larger mass has more inertia than the body having lesser mass. Thus the mass of a body is a direct measure of its inertia.

(ii) Newton's Second Law of Motion.

From every day experience we know that, if we push a body harder, it moves faster. Its velocity change in the direction of the force exerted. from such experiences it is established that when a force acts upon a certain body, the acceleration produced is proportional to the force. Symbolically it can be expressed as

$$F \propto a$$

or

$$F = ma \qquad 3.9$$

Where F is (vector) sum of all the forces acting on the body. m is the mass of the body and the equation 3.9 can be regarded as a statement of Newton's second law of motion.

The eq: 3.9 can be written as

$$\vec{a} = \frac{\vec{F}}{m}$$

m being constant it can be stated that acceleration of the body is proportional to the resultant force acting on it and the direction of acceleration is same as that of the force. It is also seen from the above equation that the acceleration for a given force is inversely proportional to the mass of the body.

The second law of motion provides us a means for the quantitative measurement of force as well as mass.

(iii) Newton's Third Law of Motion.

Newton's third law of motion can be stated as follows:

"To every action there is always an equal and opposite reaction".

When a body A exerts a force on another body B, it is called the action of the force A on B. The body B will also exert a force on body A, which will be equal in magnitude but opposite in direction. This force is called the reaction of B on A.

Let block A strike block B with a force $\vec{F}_{AB}$ and the block B will also exert a force $\vec{F}_{BA}$ on the block A which will be equal in magnitude but opposite in direction and further more the forces lie along the line joining the centres of mass of the bodies. therefore

$$\vec{F}_{A\,on\,B} = -\vec{F}_{B\,on\,A} \quad \text{and} \quad F_{A\,on\,B} = -F_{B\,on\,A}$$

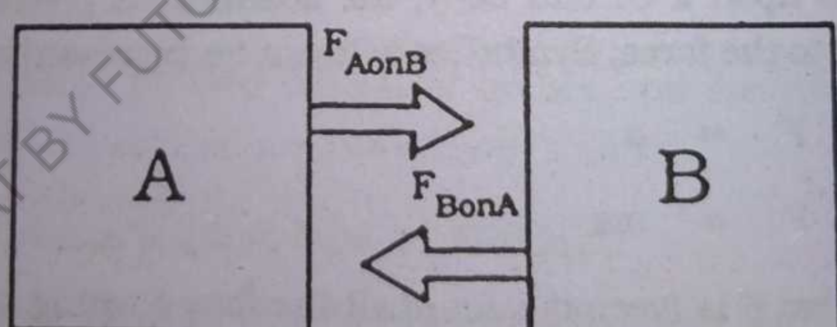


Fig. 3.6

Example 3.5

A car of mass 1000 kg travelling at a speed of 36 km/hour is brought to rest over a distance of 20 metres, find (i) average retar-

dation (ii) average retarding force.

Mass of the car = m = 1000 kg

Initial speed = V_i = 36 km/h

$$= \frac{36 \times 1000}{60 \times 60} = 10 \text{ m/s}$$

Final speed = V_f = 0 m/s

Distance covered= S = 20 metres

Now applying equation

$$V_f^2 = V_i^2 + 2aS$$

$$(0)^2 = (10)^2 + 2 \times 20a$$

$$a = \frac{-(10)^2}{2 \times 20} = -2.5 \text{ m/s}^2$$

(minus sign means retardation or deceleration)

Knowing "m" and having found "a" we now substitute in $F = ma$, to find F.

$$\text{Thus } F = ma$$

$$= 1000 \times -2.5$$

$$= -2500 \text{ N}$$

Average retardation = 2.5 m/s^2

Average retarding force = 2500 N.

3.9 MOTION OF BODIES CONNECTED BY A STRING

According to Newton's third law of motion "to every action there is always an equal and opposite reaction". This also occurs when two bodies pull each other through a material medium. Thus if we pull both ends of a string, out fingers will feel a force, this force is called the tension in the string. Similarly, when a body of weight "W" is kept suspended by a string the weight of the body pulls the string downwards, while the string pulls the body upwards with an equal force, this force is called the tension of the string. In the fig. 3.7 (a) at point B, the hand experiences a pull in the downward direction. Hence the direction of tension of the string at this point is

downward. However, at point A the string must exert a force upward to balance the weight of the body. Thus the direction of the tension at A is upward. Hence the direction of the tension depends upon the point where the string is connected. However, its magnitude remains constant at all points. When the string is not in motion, the magnitude of the tension is equal to the weight suspended from the end of the string.

Now we come across two different cases of motion of two bodies connected by a string and we shall deal with them separately.

Case I:- When both the bodies move vertically.

Consider two bodies of unequal masses m_1 and m_2 connected by a string which passes over a frictionless pulley as shown in Fig.3.7.

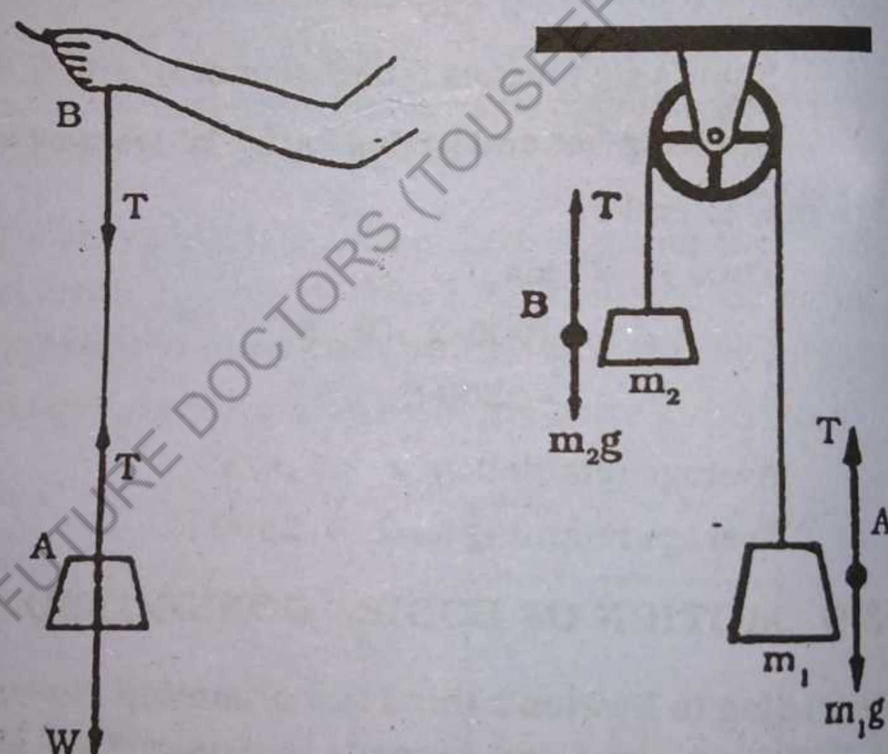


Fig.3.7 Two unequal masses are suspended by a string from a pulley in such a way that both move vertically.

Let m_1 be greater than m_2. Hence body A having greater mass will accelerate down with an acceleration say 'a', and the body B will move up with the same acceleration. Our problem is to find this acceleration "a" and the tension T in the string.

Let us first consider the motion of body A. There are two forces acting on the body (i) weight of the body $W_1 = m_1g$ acting in

the downward direction and (ii) the tension in the string which is acting in the upward direction as shown in fig 3.7 Since the body A is coming down so $W_1 > T$. Thus net force on this body is (m_1g-T) and is acting in the vertical downward direction. In fact this is the net downward force which is moving the body down with acceleration "a".

This net force also given by Newton's second law of motion is m_1a.

Thus we have the equation of motion for the body A as

$$m_1g - T = m_1a \tag{3.10}$$

Now consider the motion of body B, here also two forces are acting on B (i) the tension in the string which is acting in the upward direction and (ii) the weight W_2 of the body acting vertically downward. Since the body is moving in the upward direction so the net force acting on B in the upward direction is $T - m_2g$.

Again we can calculate the same force on block B by the application of Newton's second law of motion as m_2a.

Thus we can get the equation of motion for block B also as

$$T - m_2g = m_2a \tag{3.11}$$

For calculating "a" add equations 3.10 and 3.11

we get

$$m_1a + m_2a = m_1g - m_2g$$

$$\therefore a = \frac{m_1 - m_2}{m_1 + m_2}g \tag{3.12}$$

Tension in the string "T" can be calculated by dividing Eq: 3.10 by Eq. 3.11 as

$$\frac{m_1g - T}{T - m_2g} = \frac{m_1}{m_2}$$

By cross multipliction, we have

$$m_1m_2g - m_2T = m_1T - m_1m_2g$$

or $T(m_1 + m_2) = 2m_1m_2g$

$$\therefore \quad T = \frac{2m_1 m_2}{m_1 + m_2} g \qquad 3.13$$

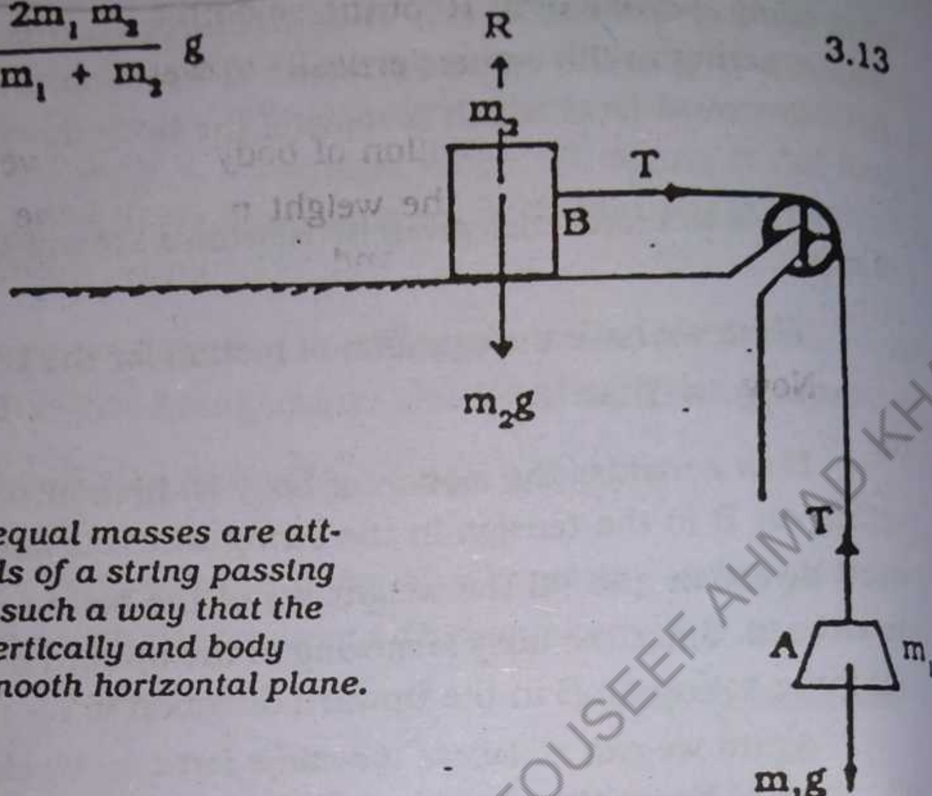


Fig. 3.8 Two unequal masses are attached to the ends of a string passing over a pulley in such a way that the body A moves vertically and body B moves on a smooth horizontal plane.

Case II:- When one body moves vertically and the other moves on a smooth horizontal surface.

Consider two bodies A and B of masses m_1 and m_2 respectively, attached to the ends of a string which passes over a pulley as shown in Fig.3.8 The pulley is frictionless and that it merely serves to change the direction of the tension in the string at that point. The body A moves vertically downward with an acceleration equal to "a" and the body B moves on a smooth horizontal surface towards the pulley with the same acceleration.

As explained in the previous case if T is the tension in the string, the downward motion of body A is governed by the equation

$$m_1g - T = m_1a \qquad 3.14$$

Now consider the motion of body B. Three forces are acting on it.

(i) The tension "T" in the string which acts horizontally towards the pulley.

(ii) The weight m_2g which acts vertically downward.

(iii) The reaction "R" of the smooth horizontal surface on the body which acts vertically upward.

Since there is no motion of body B in the vertical direction, hence the two forces i.e the weight m_2g, and the reaction of the smooth surface R are equal and opposite hence they cancel each other.

Now consider the horizontal motion of block B. If we neglect the friction, the net horizontal force acting on the block is T, the tension in the string which pulls the block towards the pulley.

Since the block is moving with acceleration "a" we can get the value of force by applying Newton's second law of motion.

$$T = m_2 a \qquad 3.15$$

For obtaining the value of "a" add eq.(3.14) and eq. (3.15) we get

$$\begin{array}{rcl} m_1g - T &=& m_1a \\ T &=& m_2a \\ \hline m_1g &=& (m_1a + m_2a) \end{array}$$

$$\text{or } (m_1 + m_2)\, a = m_1g$$

$$\text{therefore } a = \left(\frac{m_1}{m_1 + m_2}\right) g \qquad 3.16$$

Putting this value of 'a' in eq. (3.15)

we get

$$T = \left(\frac{m_1 x m_2}{m_1 + m_2}\right) g \qquad 3.17$$

Thus eq.(3.16) gives the value of acceleration while eq.(3.17) gives us the value of tension produced in the string.

Example 3.6

Two bodies A and B are attached to the ends of a string which passes over a pulley so that the two bodies hang vertically. If the mass of body A is 5kg and that of body B is 4.8 kg. Find the acceleration and tension in the string. The value of g is 9.8 m/s^2.

mass of body A = $m_1 = 5kg$.

mass of body B = $m_2 = 4.8\ kg$

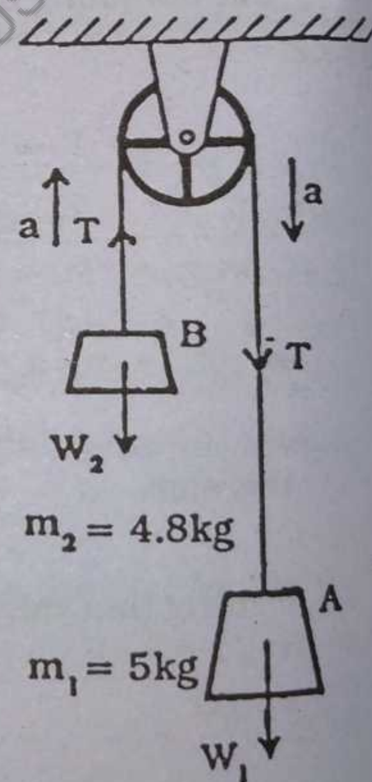
$g = -9.8\ m/sec^2$

Let acceleration of the bodies be 'a' and the tension of the string be T. In order to calculate the acceleration, we apply the following formula.

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

$$= \left(\frac{5 - 4.8}{5 + 4.8}\right) \times 9.8$$

$$= \frac{0.2}{9.8} \times 9.8 = 0.2\ m/s^2$$



For tension in the string we will apply the formula.

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right) g$$

$$= \left(\frac{2 \times 4.8 \times 5}{4.8 + 5}\right) \times 9.8$$

$$= \frac{10 \times 4.8}{9.8} \times 9.8$$

$$= 48N$$

Hence the tension in the string is 48 N and the acceleration of the bodies is 0.2 m/s^2.

3.10 MOMENTUM OF A BODY

It is commonly observed that a heavy body requires greater force to accelerate it to a given velocity than a lighter body. Similarly, greater force is required to stop a heavy body as compared to that of a lighter body within the same distance if both are moving in the same direction with the same speed.

Thus in this case we say that the body having greater mass has a greater quantity of motion than the body having a lesser mass.

Similarly, if we want to stop two bodies of the same mass within a given distance moving with different velocities, we have to apply greater force to the body moving with greater velocity than to the body moving with lesser velocity.

Thus we say that a moving body having greater velocity has a greater quantity of motion than the body having lesser velocity. This quantity of motion is known as momentum and is defined as the product of mass and its velocity.

Units of momentum

As momentum is defined as the product of mass and velocity so its units in S I System can be determined as follows

$$\text{Momentum} = \text{mass} \times \text{velocity}$$

$$= \text{Kilogram} \times \text{metre / second}$$

we get

$$\text{Momentum} = \text{Kilogram} \times \frac{\text{metre}}{\text{Second}} \times \frac{\text{Second}}{\text{Second}}$$

$$= \left(\text{Kilogram} \times \frac{\text{metre}}{(\text{Second})^2}\right) \times \text{Second}$$

$$\text{Since Kilogram} \times \frac{\text{metre}}{(\text{Second})^2} = 1 \text{ newton}$$

Therefore Momentum = newton – second

Thus the S.I unit of momentum is N-S.

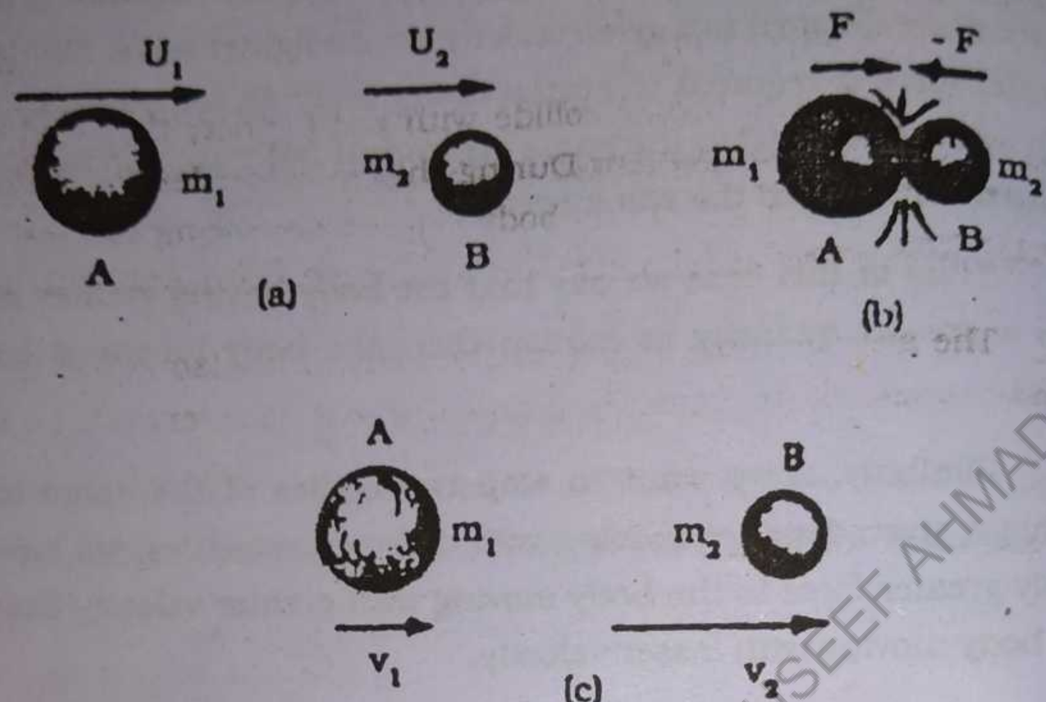


Fig 3.9 Bodies A and B (a) before collision (b) when colliding with each other and (c) after collision.

3.11 LAW OF CONSERVATION OF MOMENTUM

Suppose we have a system that is isolated; such that the constituents of the system interact with one an other and no external agency exerts a force on any of them. Truly isolated objects are not possible in the physical world, but a group of objects whose mutual interaction is much greater than their interaction with other objects can frequently be treated as if they are isolated. For example: the molecules of gas enclosed in a glass vessel at constant temperature is an isolated system of interacting bodies.

Let the system consists of two objects A and B of masses m_1 and m_2 moving with velosities U_1 and U_2 respectively, before collision and V_1 and V_2 be the velocities of the objects after collision along the same line and direction.

Thus the total momentum of the system before collision

$$= m_1U_1 + m_2U_2 \quad 3.18$$

and the total momentum of the system after collision

$$= m_1V_1 + m_2V_2 \quad 3.19$$

When the two bodies collide with each other, they come in contact for a time interval t. During this interval, let the average force exerted by the body A on body B be F. According to third law of motion, the body B will also exert a force (-F) on the body A.

The average force acting on the body B is also equal to the rate of change of its momentum during the time interval t, i.e it is equal to

$$\frac{m_2V_2 - m_2U_2}{t}$$

Similarly the average force acting upon the body A is

$$\frac{m_1V_1 - m_1U_1}{t}$$

As the forces are oppositely directed therefore

$$\frac{m_2V_2 - m_2U_2}{t} = -\frac{m_1V_1 - m_1U_1}{t}$$

or

$$(m_2V_2 - m_2U_2) = -(m_1V_1 - m_1U_1)$$

$$m_2V_2 - m_2U_2 = -m_1V_1 + m_1U_1$$

$$m_1U_1 + m_2U_2 = m_1V_1 + m_2V_2 \quad 3.20$$

This is known as law of conservation of momentum which can be stated as follows:

"If there is no external force applied to a system, then the total momentum of that system remains constant".

The above equation according to equation(3.18) and (3.19) shows that the momentum of the system before and after the colli-

sion are the same. Thus the mutual action and reaction of the bodies of an isolated system are unable to change the momentum of the system, that is, the momentum of the system is conserved. This is known as the law of conservation of momentum which can be stated as follows. "The total momentum of an isolated system of bodies is constant i.e. the total momentum of the system before and after the collision remains same"

3.12 ELASTIC COLLISION IN ONE DIMENSION

Collisions are usually classified according to whether or not kinetic energy is conserved in the collision.

An elastic collision is that in which the momentum of the system as well as the kinetic energy of the system before and after the collision is conserved i.e remains same.

In inelastic collision the momentum of the system before and after the collision changes is conserved but the kinetic energy before and after the collision changes.

When two smooth non-rotating spheres moving initially along the line joining their centres they, after having a head-on collision, move along the same straight line without rotation. Due to spherical shape, the two bodies exert forces of action and reaction during collision along the initial line of motion, so their final motion is along the same straight line

Consider two non-rotating spheres of masses m_1, and m_2 moving initially along the line joining their centres with velocities U_1 and U_2 as shown in fig.3.10. U_1 is greater than U_2 so they collide with one another and after having an elastic collision start moving with velocities V_1 and V_2 respectively in the same line and direction.

Now momentum of the system before collision $= m_1U_1 + m_2U_2$

Momentum of the system after collision $= m_1V_1 + m_2V_2$

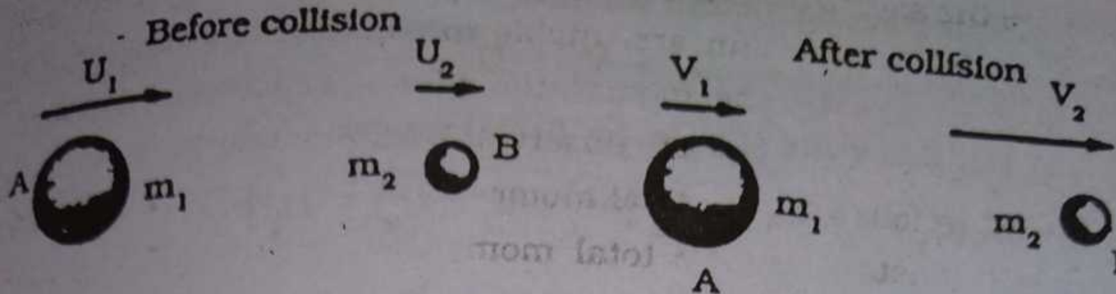


Fig. 3.10 Two spherical bodies before and after an elastic collision.

By applying the law of conservation of momentum we have

$$m_1U_1 + m_2U_2 = m_1V_1 + m_2V_2$$

or $$m_1U_1 - m_1V_1 = m_2V_2 - m_2U_2$$

or $$m_1(U_1 - V_1) = m_2(V_2 - U_2) \qquad 3.21$$

K.E of the system before collision $= \frac{1}{2}m_1U_1^2 + \frac{1}{2}m_2U_2^2$

K.E of the system after collision $= \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$

As the collision is elastic, so Kinetic energy of the system is also conserved and from the above equations we have

$$\frac{1}{2}m_1U_1^2 + \frac{1}{2}m_2U_2^2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$$

or $$m_1U_1^2 - m_1V_1^2 = m_2V_2^2 - m_2U_2^2$$

or $$m_1(U_1^2 - V_1^2) = m_2(V_2^2 - U_2^2) \qquad 3.22$$

Dividing eq. (3.22) by eq 3.21 we have

$$U_1 + V_1 = V_2 + U_2 \qquad 3.23$$

This means that the sum of initial and final velocities of the first body is equal to the sum of the initial and final velocities of second body.

Now from Eq. (3.23) we have

$$V_2 = U_1 + V_1 - U_2$$

Put this value of V_2 in eq. (3.21) we get

$$m_1(U_1 - V_1) = m_2 [(U_1 + V_1 - U_2) - U_2]$$

$$\text{or} \quad m_1U_1 - m_1V_1 = m_2U_1 + m_2V_1 - m_2U_2 - m_2U_2$$

$$\text{or} \quad m_1U_1 - m_2U_1 + 2m_2U_2 = m_1V_1 + m_2V_1$$

$$\text{or} \quad (m_1 + m_2)V_1 = (m_1 - m_2)U_1 + 2m_2U_2$$

$$\text{or} \quad V_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)U_1 + \left(\frac{2m_2}{m_1 + m_2}\right)U_2 \qquad 3.24$$

Similarly we have from eq. 3.23

$$V_1 = V_2 + U_2 - U_1$$

Putting this value in eq.3.21 we get

$$V_2 = \left(\frac{2m_1}{m_1 + m_2}\right)U_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)U_2 \qquad 3.25$$

Thus we get the values of two unknown i.e. V_1 and V_2.

There are some cases of special interest.

Case I:- If the masses of two bodies are equal, that is $m_1 = m_2 = m$ then equations (3.24) and (3.25) reduce to give $V_1 = U_2$ and $V_2 = U_1$ thus the two bodies interchange velocities after collision as shown in fig 3.11.

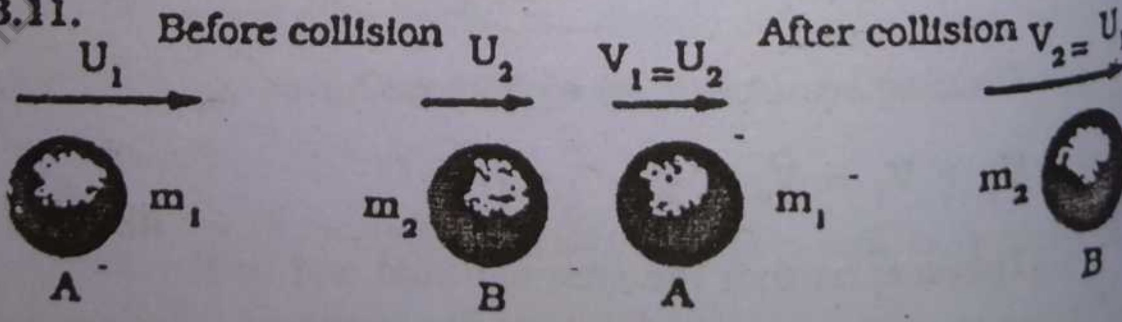


Fig. 3.11 Elastic collision between two bodies of equal masses.

<u>Case II:-</u> When the body B is initially at rest i.e $U_2 = 0$ then equations 3.24 and 3.25 give.

$$\left.\begin{aligned} V_1 &= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)U_1 \\ \text{and } V_2 &= \left(\frac{2m_1}{m_1 + m_2}\right)U_1 \end{aligned}\right\}$$

Further if $m_2 = m_1 = m$, then the first body after collision will stop and B will start moving with the velocity that A originally had as shown in Fig. 3.12.

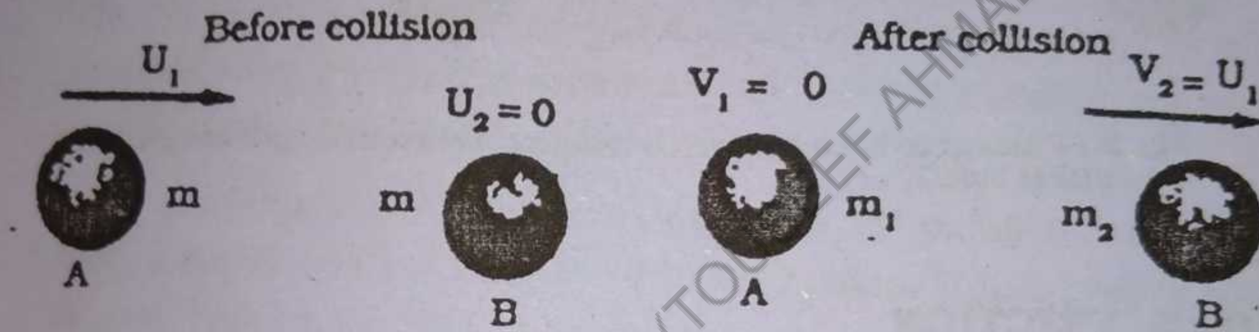


Fig. 3.12 Elastic collision between two bodies of equal masses when one of them is initially at rest.

<u>Case III:-</u> When a light body collides with a massive body at rest, then $U_2 = 0$ and $m_1 << m_2$; under these conditions m_1 is so small as compared to m_2 that it can be neglected in eq. (3.24) and eq. (3.25) and thus we have $V = - U$ and $V = 0$. Then body B will remain stationary while body A will bounce back with the velocity as shown in fig.3.13.

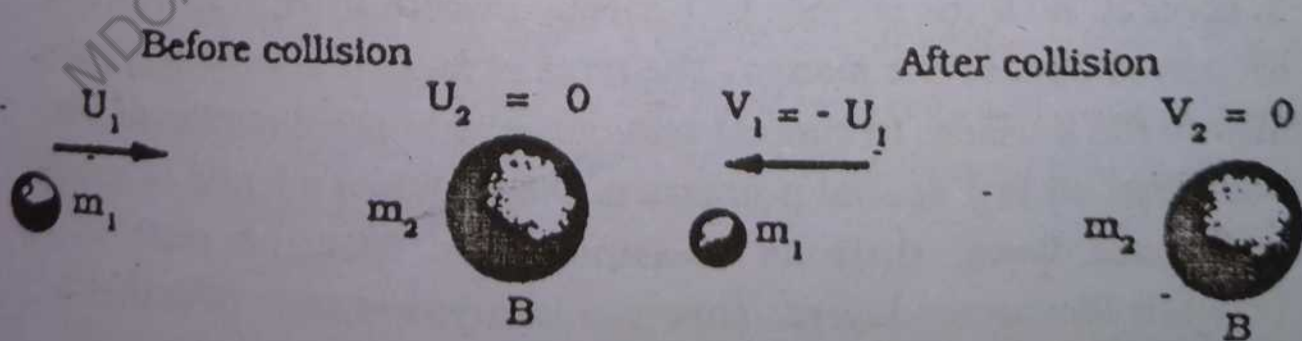


Fig. 3.13 Elastic collision between a light body and a massive body.

<u>Case IV:-</u> When a very massive body collides with a light stationary body, then $m_1 >> m_2$ and $U_2 = 0$. Now m_2 can be neglected as compared to m_1 in eq. (3.24) and (3.25). This gives $v_1 = u_1$ and $v_1 = 2u_1$. Thus after the collision, there is practically no change in the velocity of the massive body but the lighter one bounces off in the forward direction with approximately twice the velocity of the incident body as shown in fig. 3.14

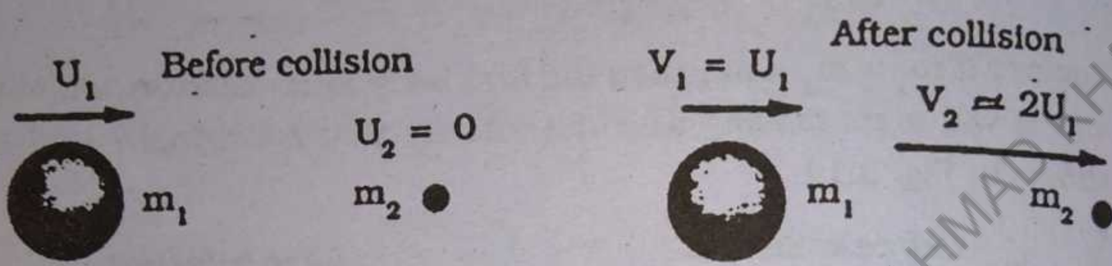


Fig. 3.14 Elastic collision between a massive body and a light body when the latter is initially at rest.

3.13 FRICTION

The surface of a solid is never perfectly smooth, consequently whenever one body slides over another, there is a sort of resistance to its motion. Hence if two bodies be in contact with each other and if we try to drag one of them over the other, a force is set up at the surface of contact, tending to resist the motion. This is called the force of friction between the surfaces in contact.

The friction is due to the roughness of the material surfaces in contact. So if the surface be perfectly smooth there is no force of friction to oppose the motion. The force of friction always acts parallel to the surfaces in contact and opposite to the direction of motion. Friction is a special property of solids. When a liquid or gaseous mass flows, there is something like frictional resistance between its various layers. This peculiar type of friction within a fluid medium is called its viscosity.

Let a rectangular solid body G as shown in fig. 3.15 remains at rest on a horizontal surface. The forces acting on G are (i) its weight mg acting vertically downwards.

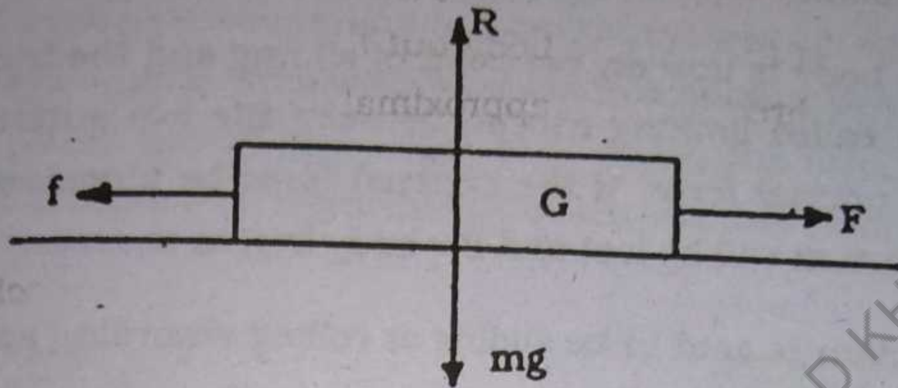


Fig: 3.15

(ii) The reaction R of the surface acting vertically upwards. In the state of rest the upward reaction R balances the weight mg and non-friction is brought into play.

If now a small force F be applied to "G" parallel to the surface, a resistance say f is offered to the motion. If this body is still at rest, it is in equilibrium under the action of the forces R, mg, f and F.

As R is equal and opposite to mg, the force "f" in this case must be equal and opposite to F.

As F is increased, f also increases. It is found that so long as F does not exceed a certain limit, there is no motion, f being thus always equal to F.

The resistance "f" which is thus brought into play by the external force "F" in a direction opposite to that of the latter is a self adjusting force and so long as the body is at rest, the force is equal to the pulling force. The force f is called the frictional force between the two bodies in contact.

Although friction is a self adjusting force, it does not however increase indefinitely with the external force.

Thus if the external force F is gradually increased, the force of friction reaches a maximum or limiting value which depends on the nature of the surfaces in contact and the magnitude of the normal reaction between them.

The body is now on the point of sliding and the friction then exerted is called limiting friction between the two surfaces under the given normal force. If the external force be increased further, the equilibrium will be lost and the body begins to move.

The friction is said to be sliding or rolling according as one body slides or rolls over the other.

Sliding friction is slightly less than the limiting friction. If, when the equilibrium is limiting, the normal reaction and the frictional force be compounded into a resultant single force, the angle which this resultant makes with the normal to the surface is called the angle of friction and the single force called the resultant reaction.

Friction plays a vital rôle in our daily life. Without friction we cannot walk, fix nails etc. Belts cling to the pulleys, drive the machinery because of friction.

Friction has both advantages as well as disadvantages. Some times we have to increase the friction e.g. sand is thrown on the uphill railway lines after rains. Similarly when the brakes of a moving car are applied, its brake shoes come in contact with the moving wheels causing an increase in friction and thus resulting in the stoppage of the car

Fig. 3.16 shows the cross section of the collar bearing in which the axle S of the revolving part is loosely fitted in the socket so as to be able to rotate. The space between the two is well lubricated.

Since rolling friction is less than the sliding friction, heavy pieces of furniture are provided with wheels at the back, which can rotate in different vertical plane.

In bicycles etc. the sliding friction is replaced by rolling friction with ball bearing arrangement in which a number of hard steel balls are placed loosely in a metal case round the axle. Fig.3.17 shows a ball bearing arrangement in which the axle S is very free to move.

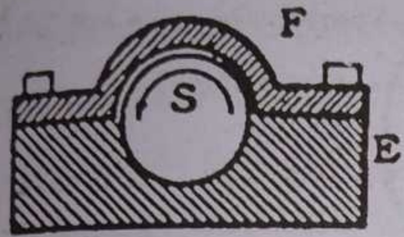


Fig. 3.16

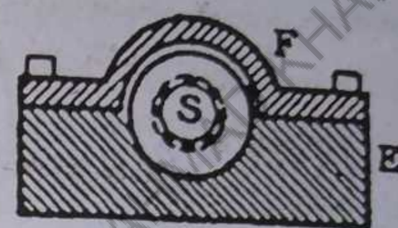


Fig. 3.17

When one body is at rest in contact with another, the friction between them is said to be static. When it is just on the point of sliding over the other, the friction is said to be limiting, and when one body is actually sliding over the other, the friction is termed kinetic or dynamic.

3.14 COEFFICIENT OF FRICTION :-

The ratio of limiting friction to the normal reaction acting between two surfaces in contact is called the coefficient of friction and is usually denoted by μ.

Thus if F be the limiting friction and R the normal reaction, then

$$\mu = \frac{F}{R} \quad \text{or} \quad F = \mu R \qquad 3.26$$

FLUID FRICTION

So far we were dealing with friction between two solids, we shall now study some thing about friction in fluids.

Bodies moving through fluids i.e. liquids or gases, experience a retarding force which is known as fluid friction or viscous drag. This is used in designing ships, aircrafts and other vehicles. In order to achieve a design in which the energy-wasting effects of the drag are reduced to a minimum, calculations are made and small-scale models are constructed which are tested in water tanks and wind tunnels.

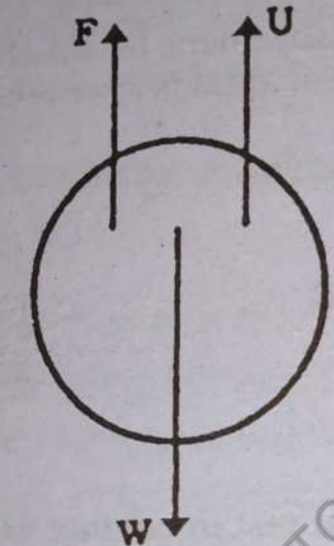


Fig. 3.18

Stokes studied the effect of viscous drag on small spheres falling through a liquid. He found that unlike bodies falling in vacuum which move with the acceleration due to gravity these spheres were found to be moving with constant velocity.

He observed that these spheres experience an upward retarding force F which is given by

$$F = 6\pi\eta r\vartheta$$

where " η " is the coefficient of viscosity

" ϑ " is the velocity of the sphere

" r " is the radius of the sphere.

Besides this there are two other forces acting on the spheres and they are:

(i) The weight of the body "W" which acts in the downward direction.

(ii) The upthrust "U" of the liquid which acts in the upward direction.

The net effect of these forces is a resultant force of magnitude (W-U) which acts in the downward direction. At constant temperature (W-U) is constant while the viscous drag F increases with velocity.

Thus if a small metal sphere is allowed to fall through a liquid, it is first accelerated so that the value of F increases and becomes equal to (W-U). at this stage the net upward and downward forces are equal and the sphere: start moving with a uniform velocity known as terminal velocity in accordance with Newton's First law of Motion.

3.15 THE INCLINED PLANE

A heavy load may be raised more easily by pulling it along an inclined surface than by lifting it vertically.

When we place a block of wood on a smooth horizontal table as shown in Fig 3.19. it remains at rest until it is pushed or pulled. Its state of rest implies that no unbalanced force is acting on it.

Fig. 3.19 Since the block is at rest, the weight must be balanced by the upward push of the table (R=w)

Let the block be placed on an inclined plane making an angle θ with the horizontal as shown in fig.3.20.(a).

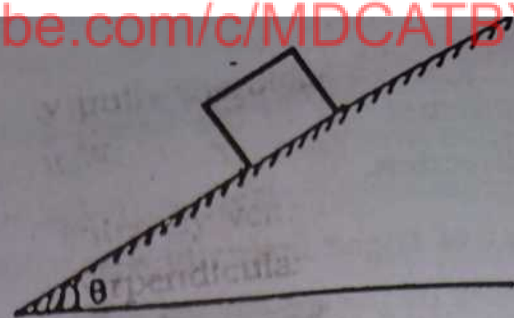


Fig. 3.20 (a)

The force of gravity pulls the block vertically downward with a force equal to its weight W.

The force is represented by vector W. The inclined plane offers a reaction which is perpendicular to the plane and is represented by R in Fig.3.20(b).

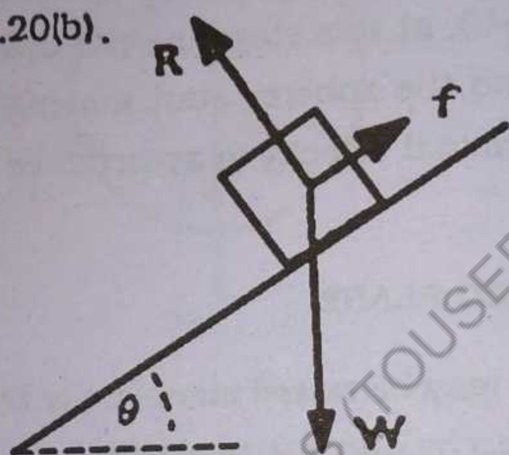


Fig. 3.20 (b)

There is also the force of friction which opposes its slipping down and is represented by "f". If the block moves (which will be the caes when frictional force is very small), it will moves down the incline. Let us take x-axis parallel to the inclined plane and Y-axis perpendicular to it. Now resolve the forces along these axes. The component of W perpendicular to the plane is W cosθ and that parallel to it is W sinθ as shown in Fig 3.20 (c).

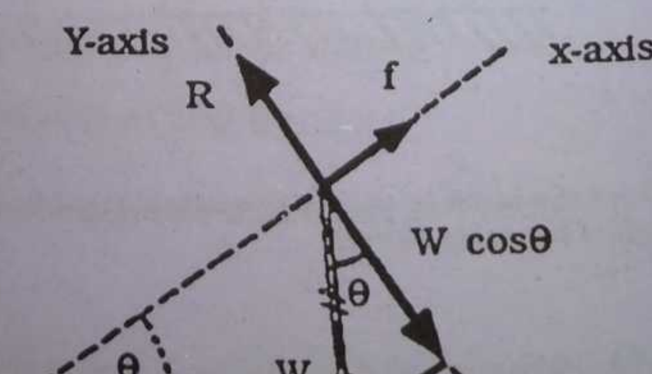


Fig. 3.20 (c)

If the block is at rest, then W sinθ acting down the plane balances the opposing frictional force. We can apply the first condition for equilibrium.

Therefore

$$\Sigma F_x = 0$$

$$\therefore f - W\sin\theta = 0$$

$$\therefore f = W\sin\theta \qquad 3.27$$

and $\Sigma F_Y = 0$

$$\therefore R - W\cos\theta = 0$$

$$\therefore R = W\cos\theta$$

If, however, the block does slide down with an acceleration a, there will be a resultant force whose magnitude is given by the product of the mass of the block and the acceleration with which it moves down.

In this case

$$W\sin\theta - f = ma \qquad 3.28$$

$W = mg$ we can write the above equation as

$$mg\sin\theta - f = ma \qquad 3.29$$

and if the force of friction is negligible, it becomes

$$mg\sin\theta = ma$$

$$\text{or } a = g\sin\theta$$

This expression is independent of the mass of the block.

PARTICULAR CASES

(i) When $\theta = 0^\circ$, $\sin\theta = 0$ and the acceleration becomes zero. This means that the block or any other body will have zero acceleration on a horizontal surface. This was the first case we discussed in this section.

(ii) When $\theta = 90°$
$\sin 90° = 1$ and hence $a = g$ (if there is no friction)

This is the case of a freely falling body.

EXAMPLE 3.7

A truck starts from rest at the top of a slope which is 1m high and 49 m long. Find its acceleration and speed at the bottom of the slope assuming that friction is negligible.

Since there is no motion perpendicular to the plane, the force R and W$\cos\theta$ must balance each other. Fig.3.21

$$\therefore \quad R - W\cos\theta = 0$$

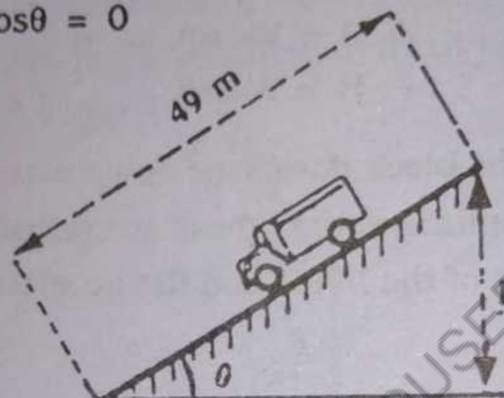


Fig. 3.21

In the absence of friction, the only unbalanced force acting on the truck is W $\sin\theta$ acting along the X-axis or parallel to the plane. If it produces an acceleration(a)and(m)be the mass of truck, then by Newton's 2nd law of motion.

$$W \sin\theta = ma$$

$$\text{but } W = mg$$

$$\therefore mg \sin\theta = ma$$

$$\text{or } a = g \sin\theta$$

$$\sin\theta = \frac{1}{49}$$

$$\therefore a = 9.8 \times \frac{1}{49} = 0.2\ m/s^2$$

To determine the speed at the bottom of the slope we make use of the equations of motion. In this problem, we have

$$V_i = 0$$

$$a = 0.2\ ms^{-2}$$

$S = 49m$

$V_f = ?$

$\therefore$ we use the equation

$$V_f^2 - V_i^2 = 2aS$$

$$V_f^2 = V_i^2 + 2aS$$

$$= (0)^2 + 2 \times 0.2 \times 49$$

$$\therefore V_f = \sqrt{19.6}$$

$$\therefore V_f = 4.4 \text{ ms}^{-1}$$

PROBLEMS

1. In an electron gun of a television set, an electron with an initial speed of 10^3 m/s enters a region where it is electrically accelerated. It emerges out of this region after 1 micro second with speed of 4×10^5 m/s. What is the maximum length of the electron gun? Calculate the acceleration.

(Ans. 0.2 metres, $399 \times 10^9 \text{m/s}^2$

2. A car is waiting at a traffic signal and when it turns green, the car starts ahead with a constant acceleration of 2 m/s^2. At the same time a bus travelling with a constant speed of 10 m/s overtakes and passes the car.
   (a) How far beyond its starting point will the car overtake the bus ?
   (b) How fast will the car be moving ?

(Ans. (a) 100 m (b) 20 m/s)

3. A helicopter is ascending at a rate of 12 m/s. At a height of 80m above the ground, a package is dropped. How long does the package take to reach the ground?

(Ans. 5.4 seconds.)

4. A boy throws a ball upward from the top of a cliff with a speed of 14.7 m/s. On the way down it just misses the thrower and fall the ground 49 metres below. Find (i) How long the ball rises? (ii) How high it goes? (iii) How long it is in air and (iv) with what velocity it strikes the ground.

(Ans. (i) 1.5 seconds (ii) 11.025m
(iii) 5 seconds (iv) 34.3 m/s)

5. A helicopter weighs 3920 newtons. Calculate the force on it if it is ascending up at a rate of $2m/s^2$. What will be force on helicopter if it is moving up with the constant speed of 4m/s.

(Ans: (i) 4720N (ii) 3920N)

6. A bullet having a mass of 0.005 kg is moving with a speed of 100 m/s. It penetrates into a bag of sand and is brought to rest after moving 25cm into the bag. Find the decelerating force on the bullet. Also calculate the time in which it is brought to rest.

(Ans. (i) 100N (ii).005 seconds.)

7. A car weighing 9800 N is moving with a speed of 40 km/h. On the application of the brakes it comes to rest after travelling a distance of 50 metres. Calculate the average retarding force.

(Ans. 1234.57 N)

8. An electron in a vacuum tube starting from rest is uniformly accelerated by an electric field so that it has a speed of 6×10^6m/s after covering a distance of 1.8 cm. Find the force acting on the electron. Take the mass of electron as 9.1×10^{-31}kg.

(Ans. 9.1×10^{-16}N)

9. Two bodies A and B are attached to the ends of a string which passes over a pulley, so that the two bodies hang ver-

tically. If the mass of the body A is 4.8 kg. Find the mass of body B which moves down with an acceleration of $0.2\ m/s^2$. The value of g can be taken as $9.8\ m/s^2$.

(Ans: 5 kg)

10. Two bodies of masses 10.2 kg and 4.5 kg are attached to the two ends of a string which passes over a pulley in such a way that the body of mass 10.2 kg lies on a smooth horizontal surface and the other body hangs vertically. Find the acceleration of the bodies, the tension of the string and also the force which the surface exerts on the body of mass 10.2 kg.

(Ans. $3\ m/s^2$, 30.6 N, 99.96 N).

11. A 100 grams bullet is fired from a 10 kg gun with a speed of 1000 m/s. What is the speed of recoil of the gun.

(Ans. 10 m/s).

12. A 50 grams bullet is fired into a 10 kg block that is suspended by a long cord so that it can swing as a pendulum. If the block is displaced so that its centre of gravity rises by 10cm, what was the speed of the bullet?

(Ans.281.4 m/s).

13. A machine gun fires 10 bullets per second into a target. Each bullet weighs 20 gm and had a speed of 1500 m/s. Find the force necessary to hold the gun in position.

(Ans. 300N)

14. A cyclist is going up a slope of 30° with a speed of 3.5 m/s. If he stops pedalling, how much distance will he move before coming to rest? (Assume the friction to be negligible).

(Ans. 1.25m).

15. The engine of a motor car moving up 45° slope with a speed of 63 km/h stops working suddenly. How far will the car move before coming to rest? (Assume the friction to be negligible.

(Ans. 22.10m)

16. In problem 15, find the distance that the car moves, if it weighs 19,600N and the frictional force is 2000N.

(Ans: 19.3m)

17. In the Figure 3.22 find the acceleration of the masses and the tension in the string.

(Ans: 0.98 m/s^2, 88.2 N)

18. Two blocks are connected as shown in fig.3.23. If the pulley and the planes on which the blocks are resting are frictionless, find the acceleration of the blocks and the tension in the string.

(Ans: 0.437 m/s^2, 223.11N)

19. Two blocks each weighing 196N rest on planes as shown in fig. 3.24. If the planes and pulleys are frictionless, find the acceleration and tension in the cord.

(Ans: 2.45 m/s^2, 49N)

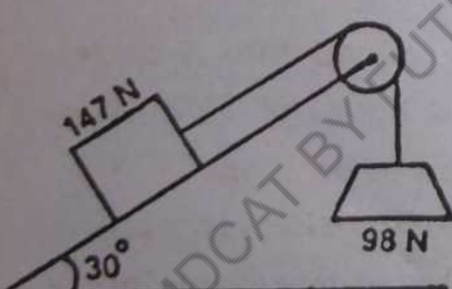


Fig. 3.22

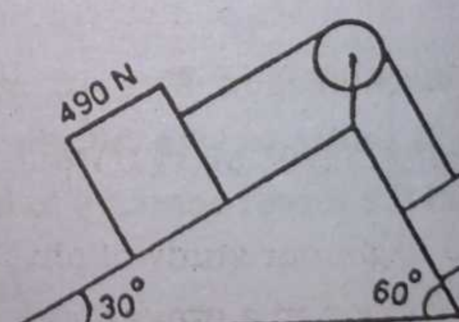


Fig. 3.23

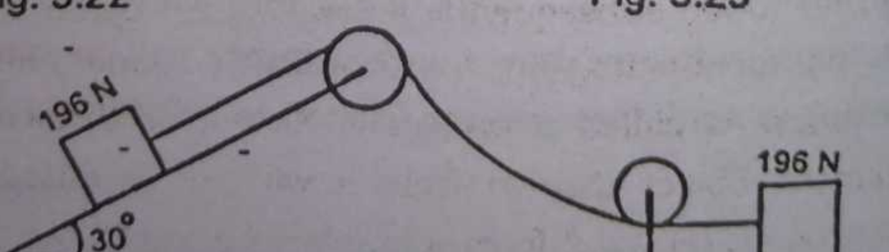


Fig. 3.24

Motion in Two Dimension

Quantitative discussions of motion are based on the measurements and calculations of positions, displacements, velocities, and accelerations. For this, we developed the equations of motion for motion with constant acceleration. The discussions were confined to one dimensional motion that is, motion along a straight line whether the line was vertical or horizontal.

If the universe were one dimensional, physics would be much simpler. But that would hardly compensate for the loss of richness of phenomena which make the physical world so beautiful and fascinating. Majority of the most important phenomena of physics simply could not take place in an one dimensional world. Thus to study various physical phenomena around us, we certainly would take to describe motion in two dimensions and ultimately in three dimensions as well. The projectile motion and circular motion are good examples of motion in two dimensions which we shall discuss here in this chapter.

4.1 PROJECTILE MOTION

Let us begin our study of physics in two dimensions by considering the motion of a projectile. Any object that is given any initial velocity and which subsequently follows a path determined by the gravitational force acting on it and by the frictional resistance of the atmosphere is called a projectile. Kicked or thrown balls, jumping animals, object thrown from a window, a missile shot from a gun, a bomb released from a bomber plane,etc., are all examples of projectiles. The path followed by a projectile is called its trajectory.

The projectile motion is surprisingly simple to analyze if the following three assumptions are made:

1. The acceleration due to gravity, $\vec{g}$, is constant over the range of motion and is directed downward.
2. The effect of air resistance is negligible.
3. The rotation of earth does not affect the motion.

This projectile motion can be analyzed by considering motion in a plane. Usually it will be vertical plane. In that case we shall use x for the horizontal coordinate and y for the vertical coordinate. It is necessary first to choose an origin, positive direction and distance scales for the coordinate axes. It is convenient to measure both the horizontal coordinate x and the vertical coordinate y of the object from its starting point. Also, we choose the positive direction of x-axis toward the right and the positive of the y-axis upward. As the object always moves downward, this choice means that the value of y will always be negative. That is, the acceleration in the y direction is -g, just as in free falls and the acceleration in x direction is zero (because air friction is neglected). In addition to this, we separate the motion in two parts, the horizontal motion along x-axis, and the vertical motion along y-axis. We are able to do this because these motions are found to be independent of one another. That is the vertical motion (motion in the y-direction) does not affect the horizontal motion (motion in the x-direction), and vice versa. Consequently, the x and y components of the displacement and velocity of an object can be calculated exactly as before if the acceleration, the initial position and velocity are known.

Suppose (i) we drop a ball from a tower, we know that the ball will undergo accelerated motion straight downward (ii) while we drop the ball, we also give it some initial velocity (say ϑ_{ox}) in the horizontal direction, obviously the motion will no longer be straight downward but will be at some angle to the vertical, as

Shown in Fig. 4.1

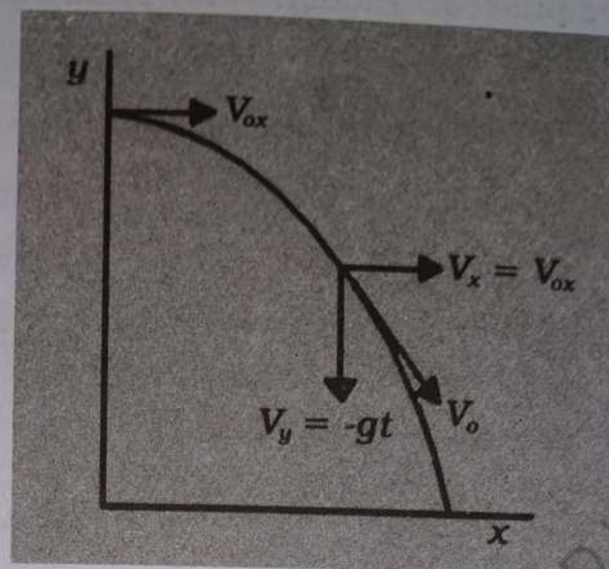


Fig: 4.1 If an object is dropped and simultaneously given an initial horizontal velocity ϑ_{oz}, this horizontal velocity component remains constant while the vertical component increases linearly with the time. Thus the motion follows a curved (actually, parabolic) path.

Let $\vec{\vartheta}_o$ represents the instantaneous velocity vector of the projectile (in this case the ball) which can be resolved into vertical component, ϑ_y , and a horizontal component, ϑ_x, as shown in Fig.4.1. Therefore, the y- component of velocity, ϑ_y, is given by

$$\vartheta_y = -gt \qquad 4.1\text{ (a)}$$

since there is no horizontal component of the acceleration, the x-component of velocity, ϑ_x, is simply given by its initial velocity.

$$\vartheta_x = \vartheta_{ox} \qquad 4.1\text{ (b)}$$

The Eq 4.1 (a) and Eq 4.1(b) are summarized by the important statement that the instantaneous velocity vector, $\vec{\vartheta}_o$, consists of two components which act independently. Only the vertical component of motion undergoes acceleration (acceleration due to gravity), whereas the horizontal component of motion proceeds at the constant initial velocity ($\vartheta_x = \vartheta_{ox}$), this means that velocity component along the x-direction never changes while the vertical

component increases linearly with time. Thus the motion of a projectile follows a curved path.

Fig. 4.2 is a stroboscopic photograph of two balls that are allowed to drop simultaneously, one of them with horizontal velocity component. The picture shows that the vertical motions in both the cases are indeed identical. However, the path followed by the projected ball (i.e the ball with initial horizontal velocity, v_{ox}) is a parabola as shown in Fig. 4.2.

addition to the initial velocity, v_{ox}, in horizontal direction, if we also allow the vertical motion to have an initial velocity, v_{oy}, then the equations which govern this motion are:

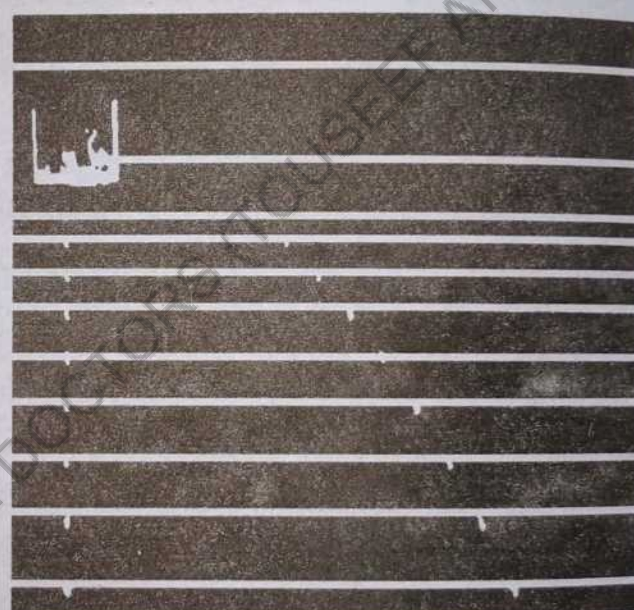


Fig. 4.2 The two balls released siumultaneously: the one the left was merely dropped while the other was given an initial horizontal velocity. The vertical components of the motion of both balls are exactly the same. The stroboscopic photograph was taken with a flash interval of 1/30s.

Horizontal motion	**(x - direction)**	
Acceleration:	$a_x = 0$	**4.2 (a)**
Velocity:	$v_x = v_{ox}$	**4.2 (b)**
Displacement	$x = v_{ox}t$	**4.2 (c)**
Vertical motion	**(y - direction)**	
Acceleration	$a_y = -g$	**4.3 (a)**
Velocity	$v_y = v_{oy} - gt$	**4.3 (b)**
Displacement	$Y = v_{oy}t - \frac{1}{2}gt^2$	**4.3 (c)**

Eq 4.2 (c) and Eq 4.3 (c) are independent description of motion one involving coordinate x and the other the coordinate y, whereas x and y both depend upon a common variable, the time t. Such equations are called parametric equations and the common variable here (t) is called the parameter.

It is not necessary that a projectile be thrown with some initial velocity in the horizontal direction. A football kicked off by a player, a missile shot from a gun, a player making a long jump, etc are all examples of projectile motion. In all these cases the bodies are projected at some angle with the horizontal. As a general case of projectile motion, we therefore consider the motion of shell shot from a gun at angle θ with the horizontal as shown in Fig.4.3.

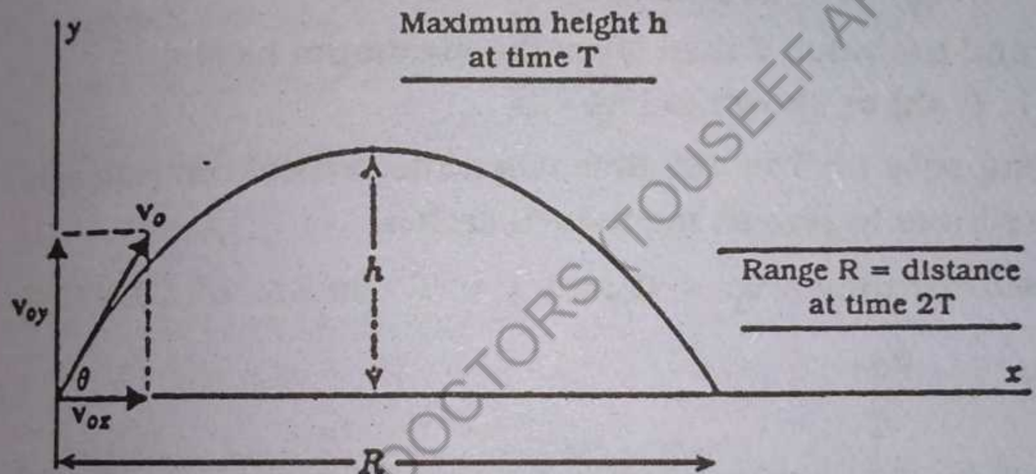


Fig. 4.3 The path of a projectile fired with an initial velocity V_o at an angle θ with respect to the horizontal.

The initial velocity $\vec{\vartheta}_o$ of the shell can be resolved into two rectangular components ϑ_{ox} and ϑ_{oy} along horizontal axis and vertical axis respectively, as shown in Fig. 4.3. The magnitudes of these components are given by

$$\vartheta_{ox} = \vartheta_o \cos\theta \qquad 4.4\ (a)$$

$$\vartheta_{oy} = \vartheta_o \sin\theta \qquad 4.4\ (b)$$

substituting for ϑ_{ox} and ϑ_{oy} in Eq 4.2 (b) and Eq 4.3 (b) respectively, the velocity components at any instant are given by

$$\vartheta_x = \vartheta_{ox} = \vartheta_o \cos\theta \qquad 4.5\ (a)$$

$$\vartheta_y = \vartheta_{oy} - gt \qquad 4.5\,(b)$$

$$\vartheta_y = \vartheta_o \sin\theta - gt \qquad 4.5\,(c)$$

These are extremely important equations and can be used to evaluate the maximum height, h, to which the projectile will rise and the overall range, R, of the projectile (shell in this case) along the horizontal surface.

4.2 MAXIMUM HEIGHT OF THE PROJECTILE

The maximum height of the projectile occurs when the vertical component of the velocity given by Eq.4.5(c) reduces to zero. That is

$$\vartheta_y = \vartheta_o \sin\theta - gt = 0$$

and the value Y then gives the maximum height, h. (Y=h) as shown in Fig. 4.3

suppose t = T be the time when the vertical component of velocity reduces to zero as mentioned earlier.

substituting $\vartheta_y = 0$ and $t = T$ in Eq. 4.5 (b), we get

$$T = \frac{\vartheta_{oy}}{g} \qquad 4.6$$

where T is half of the total time elapsed between launching and landing of the projectile. Substituting Y = h and t = T in Eq. 4.3(c), we get

$$h = \vartheta_{oy} T - \frac{1}{2} gT^2$$

substituting for T from Eq. 4.6, we find

$$h = \vartheta_{oy}\left(\frac{\vartheta_{oy}}{g}\right) - \frac{1}{2} g\left(\frac{\vartheta_{oy}}{g}\right)^2$$

$$= \frac{(\vartheta_{oy})^2}{g} - \frac{1}{-2}\,\frac{(\vartheta_{oy})^2}{-g}$$

$$= \frac{1}{2g}(\vartheta_{oy})^2$$

substituting for ϑ_{oy} from Eq. 4.4 (b), we get

$$= \frac{1}{2g} \vartheta_o^2 \sin^2\theta \qquad 4.7$$

Eq. 4.7 gives the maximum height the projectile will rise as shown in Fig. 4.3.

4.3 RANGE OF THE PROJECTILE

The horizontal distance from the origin ($x = 0, y = 0$) to the point where the projectile returns ($X = R, Y = 0$) is called the range of the projectile and is represented by R, as shown in Fig.4.3.

In order to find the range of the projectile we make use of the fact that the total flight requires a time that is twice the time necessary to reach the maximum height. Therefore we set

$X = R$: when $t = 2T$

From Eq. 4.2 c, we find

$$X = \vartheta_{ox}\, t$$

$$R = 2\,\vartheta_{ox}\, T \qquad 4.8$$

substituting for $T = \frac{\vartheta_{oy}}{g}$, we get

$$R = \frac{2}{g} \vartheta_{ox} \times \vartheta_{oy} \qquad 4.9$$

substituting for ϑ_{ox} and ϑ_{oy} from Eq. 4.5 (a), 4.5 (b), we find

$$R = \frac{2\vartheta_o^2}{g} \sin\theta \;\; \cos\theta \qquad 4.10$$

From trigonometry, we know

$$2\sin\theta\cos\theta = \sin 2\theta$$

The Eg. 4.10 can be written as

$$R = \frac{\vartheta_o^2}{g} \sin 2\theta \qquad 4.11$$

Thus the range of the projectile depends on the square of the initial velocity and sine of twice the projection angle θ.

4.4 THE MAXIMUM RANGE

The maximum range, R_{max} when the factor $\sin 2\theta$ in Eq.4.11 is maximum that is, $\sin 2\theta = 1$

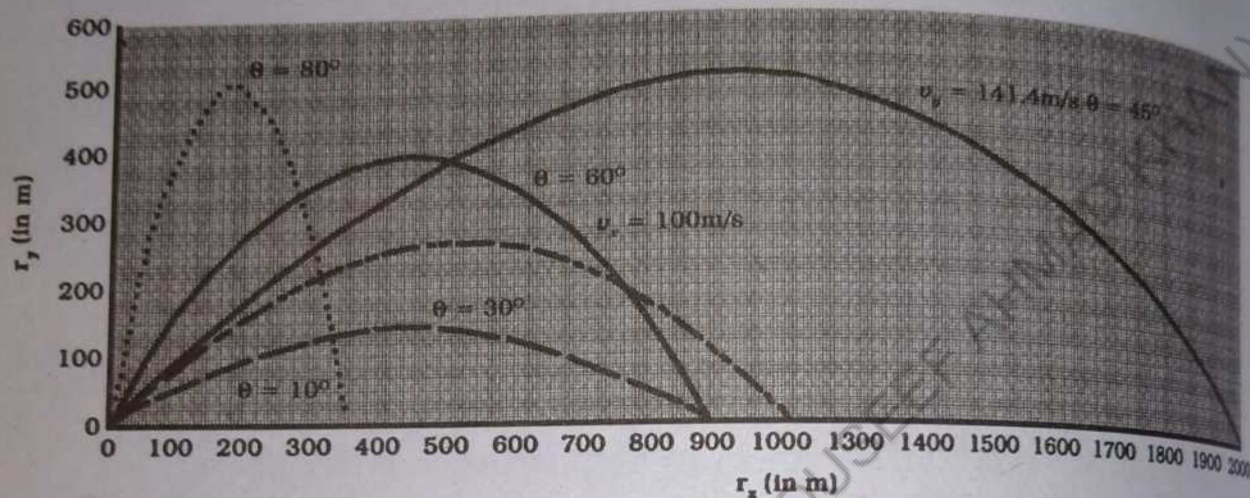


Fig. 4.4

and this happens when θ = 45° therefore the Eq.4.11 reduces to

$$R_{max} = \frac{v_o^2}{g} \; ; \text{at } \theta = 45° \qquad 4.12$$

Hence the projectile must be launched at an angle of 45° with the horizontal to attain maximum range. For all other angles greater or smaller than 45° the range will be less than R_{max} as shown in Fig.4.4.

4.5 PROJECTILE TRAJECTORY

The path followed by a projectile is referred as its trajectory. We now attempt to develop an equation which should determine the trajectory of the projectile.

The vertical displacement Y in the projectile motion is given by Eq.4.3(c)

$$Y = \vartheta_{oy}t - \frac{1}{2}gt^2$$

substituting for ϑ_{oy} from Eq 4.4 (b), we find

$$Y = \vartheta_o \sin\theta\, t - \frac{1}{2}gt^2 \qquad 4.13$$

Also from Eq. 4.2(c) and 4.4 (a)

$$X = \vartheta_{ox}\, t$$

$$t = \frac{X}{\vartheta_{ox}} = \frac{X}{\vartheta_o \cos\theta} \quad \because \vartheta_{ox} = \vartheta_o \cos\theta$$

$$t = \frac{X}{\vartheta_o \cos\theta} \qquad 4.14$$

substituting for t in Eq 4.13,

we get

$$Y = \vartheta_o \sin\theta \left(\frac{X}{\vartheta_o \cos\theta}\right) - \frac{1}{2}g\left(\frac{X}{\vartheta_o \cos\theta}\right)^2$$

$$= X\tan\theta - \left(\frac{1}{2}g\right)\frac{1}{\vartheta_o^2 \cos^2\theta}X^2 \qquad 4.15$$

For a given value of, projection angle θ and the initial velocity of the projectile, the quantities ϑ_o, sinθ, cosθ and g are constant and therefore we can lump them into another constant such that

$$a = \tan\theta \qquad 4.16$$

$$b = \frac{g}{\vartheta_o^2 \cos^2\theta} \qquad 4.17$$

The Eq. 4.15 reduces to

$$Y = aX - \frac{1}{2}bX^2 \qquad 4.18$$

Thus knowing the displacement along the vertical direction, Y, and the displacement along the horizontal direction, X, we can fix the position of the projectile at any instant, when all such

points are joined together, a trajectory of a projectile is formed. Fig.4.4 shows such trajectories that correspond to several angles $\theta = 10°, 30°, 45°, 60°, 80°$ of elevation, having same initial velocity, ϑ_o ($\vartheta_o = 100 ms^{-1}$). Note that the maximum range is attained when $\theta = 45°$. The Fig 4.4 also shows a trajectory for a projectile whose initial velocity ϑ_o is $\sqrt{2}$ times greater than its previous value (i.e $\sqrt{2} \times 100\ ms^{-1}$) with the elevation angle $\theta = 45°$, the range is twice the maximum attained by the slower projectile.

This result is in reasonable agreement with our experience in throwing balls, in spite of the fact that we ignore air resistance.

The symmetry observed in Fig 4.4 for elevation angles symmetric about 45° is due to the fact that

$$\sin[2(45° - \infty)] = \sin[2(45° + \alpha)].$$

The range will be the same for any two elevation angles $\theta = 45° \pm \alpha$ which are equal amounts greater than or less than 45°, as shown in Fig.4.4 for elevation angle 30° and 60° the small angle, of each pair produces a flat trajectory, and the large angle produces a high trajectory.

The speed ϑ of the projectile at any instant can be calculated from the components of the velocity at that instant

$$\vartheta = (\vartheta_x^2 + \vartheta_y^2)^{1/2} \qquad 4.19$$

Before we solve some numerical problems on projectile motion, we would like to summarize what we have learned so far.

(1) If air resistance is negligible, the horizontal component of velocity, ϑ_x, remains constant since there is no horizontal component of acceleration ($a_x = 0$).

(2) The vertical component of acceleration is equal to the acceleration due to gravity, g($a_y = -g$).

(3) the vertical component of velocity, ϑ_y, and the displacement in y-direction, Y, are identical to those of a freely falling body.

(4) Projectile motion can be treated as a super position of the two motions acting in the x and y directions.

4.6 APPLICATIONS

Many applications of projectile motion occur in athletics and in animals motion. Here we briefly explore some further aspect of this subject.

(1) Projectile in Athletics

The various formulas developed for the projectile motion can be directly used to analyze a tennis serve. While a player can determine his or her own best serving angle by trial and error, the projectile motion formulas can be used to predict this angle given the initial speed. The advice given in text book on tennis is sometimes based on this type of analysis. Many athletic games such as baseball, football, hockey, cricket, etc involving projectile motion that thrown, kicked, or struck can be discussed using projectile motion formulas.

(2) Horizontal jumping

Constant acceleration formulas developed in chapter 3 can be used to analyze vertical motion by animals. Similarly, to discuss horizontal motion we can use the projectile motion formula. For example, we can calculate the angle at which the jumper projects himself. The value so calculated is in close agreement with the angle seen in photographs of competitive long jumper. Using the value of initial velocity of jumper and the angle at which the jumper projects himself, we can evaluate the range.

(3) Yet in an another application of projectile motion, we know that the small angle produces flat trajectory, and the large angle a high trajectory. Air resistance tends to affect the high trajectory more because it is longer. In volley ball, cricket, a ball with high trajectory is easy to short/catch since the time of flight is so

long that the fielder has plenty of time to get into position, whereas in the case of low trajectory it is much harder to shot/catch the ball since the time of flight is not so long.

Example 4.1

A ball is kicked from ground level with a velocity of 25 ms^{-1} at an angle of 30° to the horizontal direction. (a) when does it reach the greatest height? (b) where is it at that time?

Solution

From fig 4.5 the initial velocity has components.

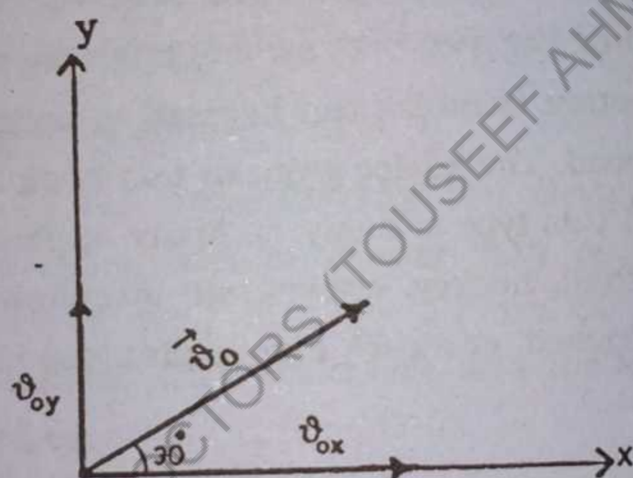


Fig. 4.5

$\vartheta_{ox} = \vartheta_o \cos 30^\circ$

$\vartheta_{ox} = (25\ ms^{-1})\ (0.866) = 21.7 ms^{-1}$

$\vartheta_{oy} = \vartheta_o \sin 30^\circ$

$\vartheta_{oy} = (25\ ms^{-1})\ (0.500) = 12.5\ ms^{-1}$

The greatest height is reached

When $\vartheta_y = 0$. Using Eq 4.3 (b)

$\vartheta_y = \vartheta_{oy} - gt$. this occurs when

$$t = \frac{(\vartheta_{oy} - \vartheta_y)}{g} = \frac{12.5\ ms^{-1} - 0}{9.8\ ms^{-2}} = 1.28s$$

$t = 1.28s$

(b) The displacements in x-direction and y- direction after

1.28s are given by Eq.4.2(c) and Eq.4.3(c) respectively.

$$X = \vartheta_{ox} t$$

$$X = (21.7ms^{-1})(1.28s) = 27.8 \text{ m}$$

$$X = 27.8 \text{ m}$$

$$Y = \vartheta_{oy}t - \frac{1}{2}gt^2$$

$$= (12.5ms^{-1})(1.28 \text{ s}) - \frac{1}{2}(9.8ms^{-2})(1.28 \text{ s})^2$$

$$Y = 7.97 \text{ m}$$

Thus the ball is 7.97m above a point on the ground which is 27.8m away from where it was kicked.

Example 4.2

A tennis ball is served horizontally from 2.4m above the ground at 30 ms^{-1}. (a) The net is 12m away and 0.9 m high. Will be ball clear the net? (b) Where will the ball land?

Solution

To find the height of the ball at the net, we must first find out the time required by ball to reach the net. From this we can then determine the height.

Solving Eq. 4.2 (c) for t

$$t = \frac{X}{\vartheta_{ox}} = \frac{12m}{30 \text{ ms}^{-1}} = 0.4s$$

$$t = 0.4s$$

Substituting t = 0.4s and $\vartheta_{oy} = 0$, the vertical displacement is

$$Y = \vartheta_{oy}t - \frac{1}{2}gt^2$$

$$= 0 - \frac{1}{2}(9.8ms^{-2})\ (0.4s)^2 = -0.78m$$

$$Y = -0.78m$$

Since the ball was intially 2.4 m above the ground, it is now (2.4 m - 0.78 m) = 1.62 m above the ground, so it easily clears the net.

(b) The ball lands when Y = - 2.4. First we have to determine the time interval, we can then find the horizontal displacement.

Substituting $\vartheta_{oy} = o$ in Eq 4.3 (c)

$$Y = \vartheta_{oy}t - \frac{1}{2}gt^2$$

$$= O - \frac{1}{2}gt^2$$

$$(t)^2 = \frac{-2Y}{g} = \frac{-2(-2.4m)}{9.8ms^{-2}} = 0.490s^2$$

$$t = 0.7s$$

The distance the ball travels horizontally before it land is given by Eq. 4.2 (c)

$$X = \vartheta_{ox}t = (30ms^{-1})\ (0.7s) = 21.0m$$

$$X = 21.0m$$

Example 4.3

An artillery piece is pointed upward at an angle of 35° with respect to the horizontal and fires a projectile with a muzzle velocity of 200 m s^{-1}. If air resistance is negligible, to what height will the projectile rise and what will be its range?

The height is given by Eq. 4.7-

$$h = \frac{1}{2}\frac{(\vartheta_{oy})^2}{g} = \frac{1}{2}\frac{(\vartheta_o)^2}{g}\sin^2\theta$$

$$h = \frac{1}{2}\ \frac{(200ms^{-1})^2}{9.8\ ms^{-2}} \times \sin^2 35^\circ$$

$$h = 672.4m$$

The range is given by Eq. 4.11

$$R = \frac{\vartheta_o^2}{g}\ \sin 2\theta$$

$$R = \frac{(200ms^{-1})^2}{9.8\ ms^{-2}} \times \sin 70^\circ$$

$$R = 3835.48m$$

A rifle bullet fired with the same initial conditions would not travel nearly this far. Because a rifle bullet has a much larger surface to mass ratio than does an artillery shell, air resistance effect is much more severe and drastically reduces the range.

Example 4.4

A player throws a ball at an initial velocity of $36ms^{-1}$. (a) Calculate the maximum distance the ball can reach, assuming the ball is caught at the same height at which it was released. (b) If he wishes to throw the ball half the maximum distance in the shortest possible time, compute the angle of elevation in this case. (c) What are the elapsed times in the two cases?

Solution

(a) The maximum range occurs for an elevation angle of 45° and the maximum range can be calculated using Eq. 4 12 Thus

$$R_{max} = \vartheta_o^2/g = (36ms^{-1})^2\ /\ (9.8ms^{-2}) = 132m$$

$$R_{max} = 132m$$

(b) Now

$$R = \frac{R_{max}}{2} = \frac{132m}{2} = 66\ m$$

Solving Eq 4.11 for $\sin 2\theta$, using $R = 66m$,

$$\sin 2\theta = g\,R/\vartheta_o^2$$

$$= (9.8\ ms^{-2})\ (66\ m)\ /\ (36\ ms^{-1})^2 = 0.5$$

$$\sin 2\theta = 0.5$$

$$\theta = 15^\circ$$

Thus the ball thrown at 15° elevation angle will cover half of the maximum range. The same range can be obtained with an elevation angle of 75°, but the elapsed time will be longer due to different trajectory.

(c) The time elapsed in the above two cases can be calculated by using Eq.4.6, and doubling the result, since the T represents half of the total time elapsed between launching and landing.

The times are

$$(T_1)\ \text{first case} = \frac{2\vartheta_{oy}}{g} = \frac{2\vartheta_o \sin 45^\circ}{g}$$

$$= 2(36ms^{-1})\ (\sin 45^\circ)\ /\ (9.8ms^{-2})$$

$$(T_1)\ \text{first case} = 5.2\ s$$

$$(T_2)\ \text{Second Case} = \frac{2\vartheta_{oy}}{g} = \frac{2\vartheta_o \sin 15^\circ}{g}$$

$$= 2(36ms^{-1})\ (\sin 15^\circ)\ /\ (9.8ms^{-2})$$

$$(T_2)\ \text{Second Case} = 1.90\ s$$

Notice that the elapsed time in case (b) is less than half in (a), even though the range is halved, because the trajectory is, much flatter.

4.7 UNIFORM CIRCULAR MOTION -

In secondary classes you have learnt about uniform circulation motion up to some extent. In this chapter the subject is intro-

duced in a bit elaborate form to impart sufficient knowledge about the subject and its applications.

In preceding article we have discussed the projectile motion which was a case of two dimensional motion. Another very important case of two dimensional motion is that of motion in a circular path. For example, rotation of earth around the sun, the rotation of moon around the earth, the spinning of earth about its own axis, artificial satellite orbiting the earth, lawn mover blades, automobile wheel, a fly wheel rotating about an axis, etc., are very common examples of circular motion. One important consideration of this motion is that each point in such an object is under-going circular motion.

When an object such as P in Fig.4.6 moves along a circular path in such a way that its speed is uniform, that is the magnitude, ϑ, of its velocity, $\vec{\vartheta}$, is constant. This type of motion is known as uniform circular motion. To describe the uniform circular motion we would like to define the following :

(1) Angular displacement

(2) Angular velocity/angular frequency

(3) Period of circular motion.

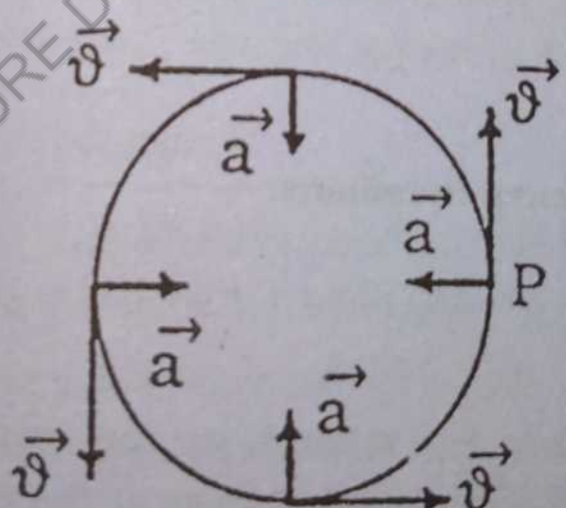


Fig. 4.6 Velocity and acceleration of a particle in uniform circular motion

4.8 ANGULAR DISPLACEMENT

Consider an object moving along a circular path of radius r

as shown in Fig. 4.7. Consider further that the object initially is at the point P_1 on the circumference of the circle. After a small interval of time, it moves to the position P_2. Evidently angle P_1OP_2 or θ represents the angular displacement of the object. The angular displacement is measured in degrees. However, it is more convenient to measure angles in another unit called the radian.

The length, s, of an arc on a circle Fig. 4.7 is directly proportional to the radius, r, of the circle and to the angle θ subtended by the ends of the arc. One radian is defined to be the angle subtended where the arc length, s, is exactly equal to the radius of the circle. Thus straightaway we can write

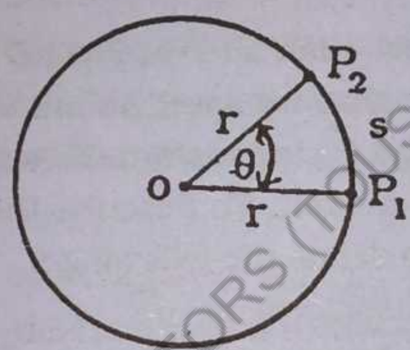


Fig: 4.7 The position of an object moving in circular path

$$\theta = \frac{S}{r} \qquad 4.20$$

where θ is measured in radians.

Also

$$S = r\,\theta \qquad 4.21$$

When θ is measured in radians, we can easily calculate the length of an arc which subtends this angle at the centre of the circle. For one complete revolution, $\theta = 360°$, then the arc length s becomes the circumference of the circle, that is,

$$S = 2\pi r \qquad 4.22$$

Comparing Eq. 4.21 and Eq. 4.22, we write,

$$r\theta = 2\pi r$$

$$\theta = 2\pi \quad \text{radians} \tag{4.23}$$

$$\text{or } \theta = 360^\circ = 2\pi \quad \text{radians} \tag{4.24(a)}$$

therefore 4.24(b)

$$1 \text{ rad.} = \frac{360^\circ}{2\pi} = 57.2958^\circ \equiv 57.3^\circ$$

also

$$1^\circ = \frac{2\pi}{360^\circ} = 0.01745 \text{ rad}$$

Note that the measure of an angle whether in degrees or radians does not have physical dimensions of length, mass, or time since it is the ratio of two lengths. Although we carry the unit radian abbreviated rad through our calculations to remind us that angles are being measured in radians, however, this unit does not appear in the final answer. For example, the length of arc s on a circle of radius 0.15 m which is subtended by angle 0.5 rad., then

$$S = r\theta = (0.15 \text{ m})(0.5 \text{ rad}) = 0.075 \text{ m}$$

Thus the unit rad. does not appear in the final answer

4.9 Angular velocity

Suppose a body P moves counter clockwise in a circle of radius r as shown in Fig. 4.8. The angular position of P is θ_1 at a time $t = t_1$ and at a later time $t = t_2$ its angular position is θ_2 with respect to the x-axis as shown in Fig. 4.8.

The angular displacement $= \theta_2 - \theta_1 = \Delta\theta$

The time interval $= t_2 - t_1 = \Delta t$

We define the average angular speed of the particle P. ω_{av} (Greek letter "Omega") in the time interval Δt as the ratio of the an

gular displacement $\Delta\theta$ to Δt:

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} \qquad 4.25$$

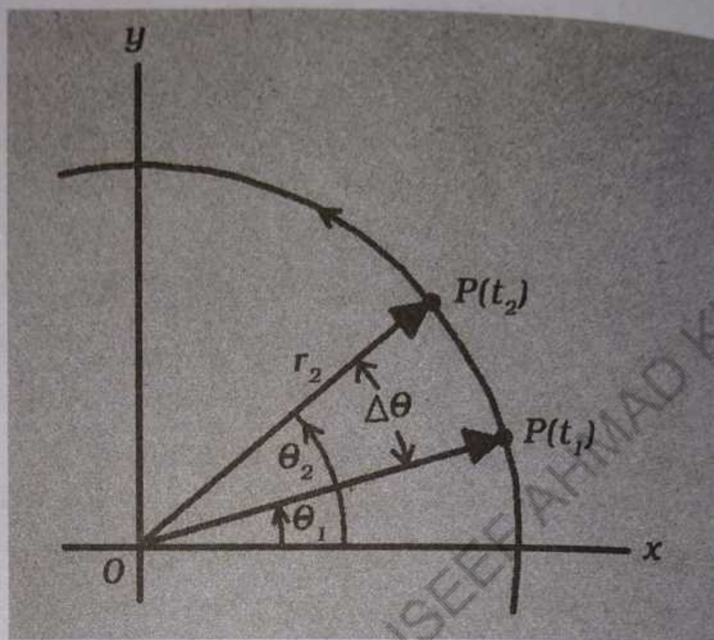


Fig. 4.8

The Eq 4.25 gives the magnitude of the average angular velocity.

The instantaneous angular speed, ω_{ins} is defined as the limit of this ratio as Δt approaches zero:

$$\omega_{ins} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \qquad 4.26$$

The Eq 4.26 gives the magnitude of instantaneous angular velocity. If the angle θ is measured in radians, the unit of angular velocity is then radian per second (rad s^{-1})

Also

1 radian per second = 1 rad s^{-1} = 1 s^{-1}

the rad does not appear in the final answer. Other units such as revolution per minute (rev. min^{-1}), are also in common use

$$1 \text{ rpm} = \left(\frac{2\pi}{60}\right) \text{rad.s}^{-1}$$

Because we know that the unit s^{-1} is the unit of frequency, therefore, it is equally appropriate to refer to 'ω' as angular frequency.

It is important to recognize that points at different radial distances on a rotating body have different linear speeds along their circular path since the displacements are different. However, every point on a rotating body has the same value of angular velocity since all radial lines fixed in the body perpendicular to the axis of rotation rotates simultaneously through the same angle in the same time. Thus the angular velocity is characteristic of the rotating body as a whole. By definition the angular velocity depends upon the rate of change of the angular displacement, therefore in circular/rotational motion the angular displacement rather than displacement, is the basic quantity to be measured.

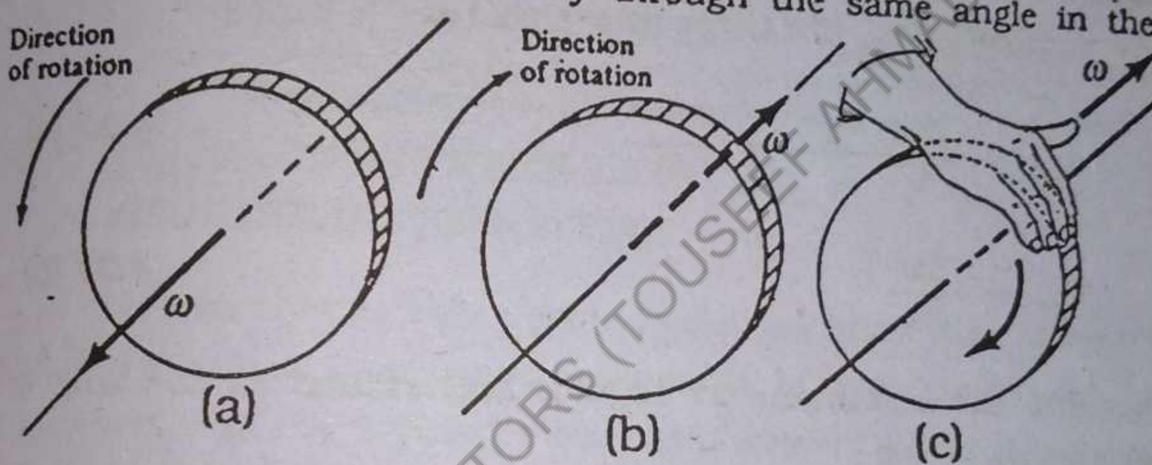


Fig. 4.9 (a) For counter clockwise rotations $\vec{\omega}$ is directed out of the page.
(b) $\vec{\omega}$ is directed into the page for clockwise.
(c) Curling the fingers of the right hand in the direction of rotation, the thumb points perpendicular to the disk in the direction of $\vec{\omega}$.

The angular velocity vector, $\overrightarrow{\omega}$, is conventionally taken to be directed along the axis of rotation. It is directed out of the page, parallel to the axis of rotation, if the rotation is counter clockwise as shown in Fig-4.9(a). If the rotation is clockwise, as in Fig 4.9(b), $\overrightarrow{\omega}$ is directed into the page. One way of assigning the direction of the angular velocity vector, $\overrightarrow{\omega}$, is to curl the fingers of right hand

around the axis of rotation: in the direction of rotation. The right hand thumb then points in the direction of $\vec{\omega}$ as shown in Fig.4.9(c)

4.10 ANGULAR ACCELERATION

When the angular velocity changes with respect to time, an angular acceleration is produced. That is, the rate of change of angular velocity with respect to time defines angular acceleration.

Let ω_1 and ω_2 be the magnitudes of instantaneous angular velocities at time t_1 and t_2 respectively. We define the average angular acceleration α_{av} (the Greek letter alpha) as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad 4.27\ (a)$$

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t} \qquad 4.27\ (b)$$

and the instantaneous angular acceleration as the limit of this ratio as $\Delta t \rightarrow 0$

$$\alpha_{ins} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} \qquad 4.28$$

The S.I units of angular acceleration is radian per second per second or rad. s^{-2}.

The angular acceleration vector $\vec{\alpha}$ points along the axis of rotation and is either parallel or opposite to the vector $\vec{\omega}$. For example. if we increase the rate of rotation of a disc (that, more revolutions/rotation per second) then the angular acceleration vector, $\vec{\alpha}$, is directed parallel to the angular velocity vector, $\vec{\omega}$, shown in Fig 4.10 (a). If we decrease the rotation rate (less number of revolutions/rotation per second) then the angular acceleration vector, $\vec{\alpha}$, is directed opposite to the angular velocity vector, $\vec{\omega}$, as shown in Fig 4.10 (b).

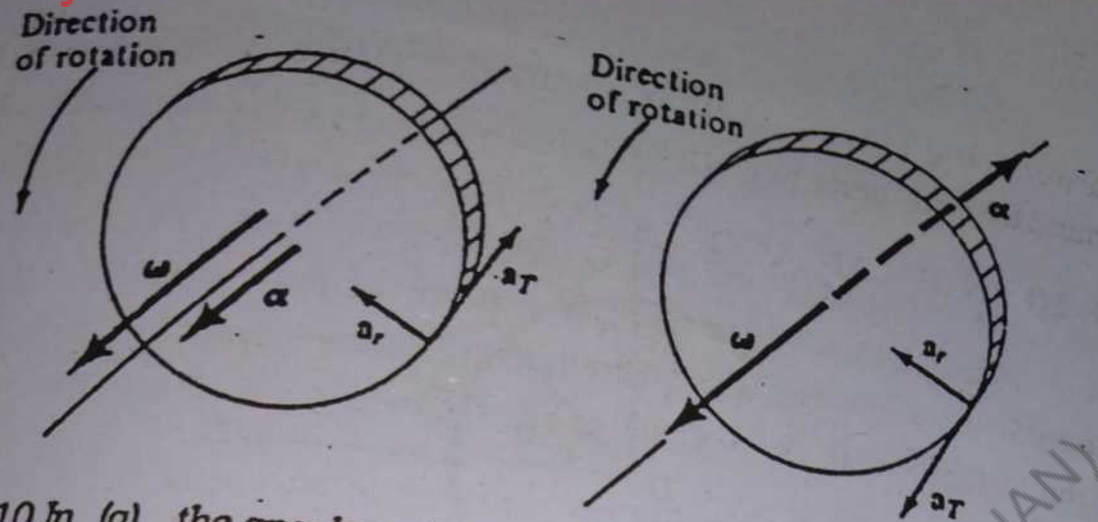


Fig. 4.10 In (a) the angular velocity of the disk is increasing so $\vec{a}$ and $\vec{w}$ are parallel In.
(b) the angular velocity is decreasing so $\vec{a}$ and $\vec{w}$ are antiparallel. The tangential and radial acceleration a_r and a_T of a points on the disk are also shown.

4.11 RELATION BETWEEN ANGULAR AND LINEAR QUANTITIES.

Here we shall establish some useful and interesting relations between the linear velocity and acceleration of an arbitrary point in the object, and the angular velocity and acceleration of a rotating object. In doing so once again we should bear in our mind the fact that when an object rotates about a fixed axis, every point in the object moves in a circle whose centre is on the axis of rotation.

Consider a particle P in an object (in x-y plane) rotating along a circular path of radius r about an axis through 0, perpendicular to the plane of the figure (the z-axis) as shown in Fig 4.11.

Suppose the particle P rotates through an angle Δθ, in a time Δt Using Eq. 4.20, we find

$$\Delta\theta = \frac{\Delta s}{r} \qquad 4.29$$

dividing both sides of Eq.4.29 by Δt– the time duration in which rotation occurred, we get

$$\frac{\Delta\theta}{\Delta t} = \frac{1}{r}\ \frac{\Delta s}{\Delta t} \qquad 4.30\ (a)$$

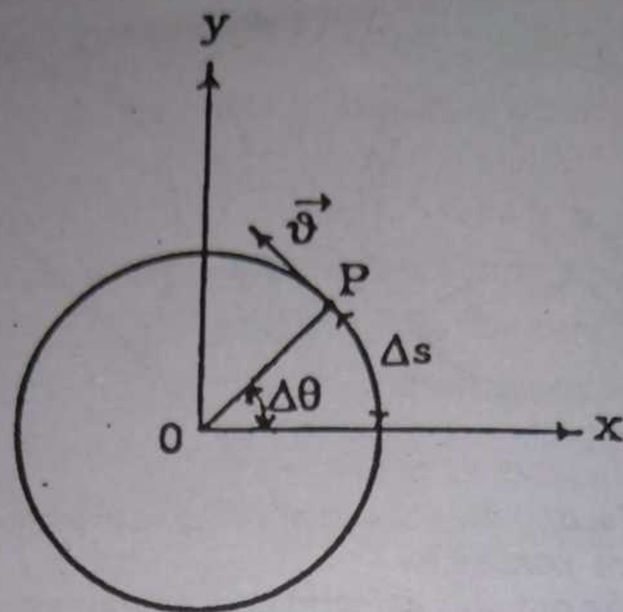


Fig 4.11 Rotation of a particle about an axis through O. perpendicular to the plane of the figure (the z axis). Note that a point P rotates in a circle of radius r centered at O.

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta \theta}{\Delta t} \qquad 4.30\text{ (b)}$$

If the time interval Δt is very small ($\Delta t \to 0$) then the angle through which the particle P moves is also very small and therefore the ratio $\frac{\Delta\theta}{\Delta t}$ gives the instantaneous angular speed, ω_{ins} as before. Also, when Δt is very small($\Delta t \to 0$), Δs is very small, and the ratio $\Delta s/\Delta t$ gives instantaneous linear speed, ϑ_{ins}. Therefore the Eq. 4.30 can be written as

$$\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = r\lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t} \qquad 4.31$$

Now by definition

$$\vartheta = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t}$$

therefore, the Eq. 4.31 reduces to

$$\vartheta = r\omega \qquad 4.32$$

4.12 TANGENTIAL VELOCITY

The distance Δs is traversed along an arc of the circular path followed by the particle P as it rotates during the time Δt. Thus ϑ must be the linear velocity of the particle lying along the arc, a velocity that is tangent to the circular path as shown in Fig 4.10. Due to this reason the linear velocity is often referred to as the tangential velocity of a particle moving along a circular path, and is written as

$$\vec{V_t} = \vec{\omega} \times \vec{r} \qquad 4.33(a)$$

The tangential velocity ϑ_t of a particle moving in a circular path is given by the product of the distance of the particle from the axis of rotation and the angular velocity.

The Eq 4.33 (a) gives an important result that every point on the rotating object has same angular velocity whereas the linear velocity/tangential velocity is not same for every point on the rotating object. The Eq.4.33(a) also shows that the tangential velocity of a point on the rotating object increases as we move outward from the centre of rotation i.e, as r increases. Eq.4.33(a) has been derived using the equation which defines radian, hence the equation is valid only when the angular speed, ω, of the rotating object is measured in radians per unit time. Other measures of the angular speed, ω, such as revolutions per second or degrees per second cannot be used.

Suppose an object rotating about a fixed axis, changes its angular velocity by $\Delta\omega$ in a time Δt. Then the change in tangential velocity, $\Delta\vartheta_t$, at the end of this interval is

$$\Delta\vartheta_t = r\,\Delta\omega \qquad 4.33\text{(b)}$$

dividing both sides by Δt, we get

$$\frac{\Delta\vartheta_t}{\Delta t} = r\frac{\Delta\omega}{\Delta t} \qquad 4.34$$

If the time interval is very small ($\Delta t \longrightarrow 0$), the Eq 4.34 can be written as

$$\lim_{\Delta t \to 0} \frac{\Delta v_t}{\Delta t} = \lim_{\Delta t \to 0} r \frac{\Delta \omega}{\Delta t}$$

Using Eq 4.28, tangential and the definition of instantaneous linear / tangential acceleration, we write

$$a_t = r\alpha \qquad 4.35$$

Thus the tangential acceleration of a point on a rotating object is product of the distance of the point from the axis of rotation and the angular acceleration.

4.13 THE PERIOD

The time required for one complete revolution or cycle of the motion is called time period. The period is denoted by T. We know that greater the angular velocity, the shorter the time required to make a revolution or vice versa. Thus the angular speed, ω, and the time period, T, are inversely related. Therefore

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi f} = \frac{1}{f} \qquad 4.36.$$

4.14 CENTRIPETAL ACCELERATION

Suppose an object moves without acceleration. This means there is no change in the velocity of the object. In other words magnitude and direction of the velocity vector remain constant. Conversely, if there is any change in the velocity vector, then there must be acceleration. The change in the velocity vector is either due to change in its magnitude or change in its direction. It is the latter situation that is occurring for an object moving in a circular path with constant speed. Thus, an object moving in a circular path with uniform speed is continually accelerated. We shall now show that the acceleration vector in this case is perpendicular to the circular path and always points toward the centre of

circle. Because the acceleration is always directed toward the centre of the circle, it is called centripetal acceleration; the word centripetal is derived from two Greek words meaning "seeking the centre". Thus, the acceleration produced by virtue of the changing direction of the velocity of an object moving in a circular path is called centripetal acceleration, $\vec{a}_c$.

Some times the centripetal acceleration, $\vec{a}_c$ is denoted by $\vec{a}_\perp$ indicating that this acceleration acts perpendicular to the path. We shall now show that the magnitude, a_c, of the centripetal acceleration, $\vec{a}$, is $\frac{\vartheta^2}{r}$ and its direction is always toward the centre of the circle.

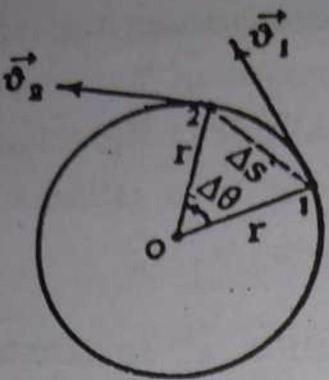
In order to calculate the magnitude, a_c, of the centripetal acceleration, $\vec{a}_c$, we must first find the velocity difference, $\vec{\Delta\vartheta}$ for two successive positions of an object moving along a circular path, say at time $t = t_1$ and $t = t_2$. Suppose the object takes a time $\Delta t = t_2 - t_1$ to go from position 1 to position 2, as shown in Fig 4.12 (a).

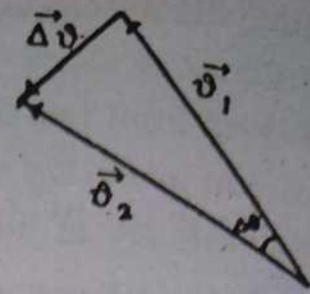
Let at time t_1 the velocity vector of the moving object be $\vec{\vartheta}_1$. At time t_2 the motion has progressed by an angle $\Delta\theta$ and the velocity vector at position 2 is $\vec{\vartheta}_2$ as shown in Fig 4.12(a). For uniform circular motion $\vartheta_1 = \vartheta_2 = \vartheta$ but the velocity vectors $\vec{\vartheta}_1$ and $\vec{\vartheta}_2$ are different. Thus

$$\Delta\vec{\vartheta} = \vec{\vartheta}_2 - \vec{\vartheta}_1 \qquad 4.37$$

The vector difference, $\vec{\Delta\vartheta}$ is solely due to the different directions of the velocity vectors at the two positions. If there is no change in the direction of the velocity vectors then the vector difference, $\Delta\vec{\vartheta}$, vanishes. The vector difference between two velocity vector is sketched in vector diagram as in Fig 4.12(b).

Note the angle $\Delta\theta$ between the velocity vector $\vec{\vartheta}_1$ and $\vec{\vartheta}_2$ is the same as $\Delta\theta$ in Fig 4.12(a), since the velocity vectors $\vec{\vartheta}_1$ and $\vec{\vartheta}_2$ are





$$\vec{\vartheta}_2 = \vec{\vartheta}_1 + \vec{\Delta\vartheta}$$

$$\vec{\Delta\vartheta} = \vec{\vartheta}_2 - \vec{\vartheta}_1$$

Fig 4.12 (a) (b)

each perpendicular to radius lines at position 1 and at position 2,respectively. It follows from geometry that the triangle formed by the two radial lines and Δs (Fig 4.12(a))is similar to the triangle formed by the vectors $\vec{\vartheta}_1$, $\vec{\vartheta}_2$ and $\vec{\Delta\vartheta}$ (Fig 4.12(b)), since both are isosceles triangles, and the angles $\Delta\theta$ are the same. Hence.

$$\frac{\Delta\vartheta}{\vartheta} = \frac{\Delta s}{r} \qquad 4.38\text{ (a)}$$

$$\Delta\vartheta = \vartheta\frac{\Delta s}{r} \qquad 4.38\text{ (b)}$$

Where Δs is straight line distance between the position 1 and the position 2 as shown in Fig 4.12 (a). Dividing both sides of Eq.4.38(b) by Δt, we find

$$\frac{\Delta\vartheta}{\Delta t} = \frac{\vartheta}{r}\frac{\Delta s}{\Delta t} \qquad 4.39$$

When Δt is very small ($\Delta t \rightarrow 0$)

$$\lim_{\Delta t\rightarrow 0}\frac{\Delta\vartheta}{\Delta t} = \frac{\vartheta}{r}\lim_{\Delta t\rightarrow 0}\frac{\Delta s}{\Delta t} \qquad 4.40$$

Substituting

$$\vartheta = \lim_{\Delta t\rightarrow 0}\frac{\Delta s}{\Delta t}$$

$$a_c = \lim_{\Delta t\rightarrow 0}\frac{\Delta\vartheta}{\Delta t}$$

The Eq. 4.40 reduces to

$$a_c = \frac{\vartheta^2}{r} \qquad 4.41$$

Eq. 4.41gives the magnitude of the centripetal acceleration.

In Fig 4.12 (b) when Δt is very small ($\Delta t \rightarrow 0$), Δs and $\Delta\theta$ are also very small. In this situation, the $\vec{\vartheta}_2$ will be parallel to $\vec{\vartheta}_1$ and the vector $\Delta\vec{\vartheta}$ will be approximately perpendicular to them, pointing toward the centre of the circle. Since the direction of the acceleration vector, $\vec{a}_c$, is same as the direction of $\Delta\vec{\vartheta}$, the vector $\vec{a}_c$ always points toward the centre of the circle.

From Eq 4.32

$$\vartheta = r\omega$$

Solving for ϑ, the Eq 4.41 reduces to

$$\alpha_c = r\omega^2$$

$$= r\left(\frac{2\pi}{T}\right)^2 \qquad \because \omega = \frac{2\pi}{T}$$

$$\alpha_c = \frac{4\pi^2 r}{T^2} \qquad 4.42$$

In order to understand the difference between the centripetal acceleration, $\vec{a}_c$, and the tangential acceleration, $\vec{a}_t$, we consider an object moving in a circular path. If the object is moving, it always has centripetal component of acceleration, because the direction of travel of the object and hence the direction of its velocity is continuously changing. If the speed of the object is increasing or decreasing (the speed is not constant or motion is not uniform) it also has a tangential component of acceleration. That is, the tangential component of acceleration arises when the speed of the object is changed; the centripetal component of acceleration arises when the direction of motion is changed. When both components of acceleration exist simultaneously, the tangential acceleration

component is tangent to the circular path whereas the centripetal acceleration always directed toward the centre of the circular path as shown in Fig 4.13 (a). The centripetal acceleration, $\vec{a}_c$, and the tangential acceleration, $\vec{a}_t$, are also represented by $\vec{a}_\perp$, and $\vec{a}_\parallel$, respectively, since the former acts perpendicular to the instantaneous velocity and the latter acts along the direction of the velocity.

These two components of acceleration are perpendicular to each other, then total acceleration, $\vec{a}$, by using vector diagram Fig 4.13 (b), is given by

$$\vec{a} = \vec{a}_c + \vec{a}_t \qquad 4.43$$

The magnitude, a, of the total acceleration, $\vec{a}$, is

$$a = \sqrt{a_c^2 + a_t^2} \qquad 4.44$$

The direction of $\vec{a}$ with respect to $\vec{a}_c$ is given by

$$\phi = \tan^{-1}\left(\frac{a_t}{a_c}\right) \qquad 4.45$$

where $\vec{a}_t$ and $\vec{a}_c$ represent magnitude of the tangential and the centripetal acceleration respectively.

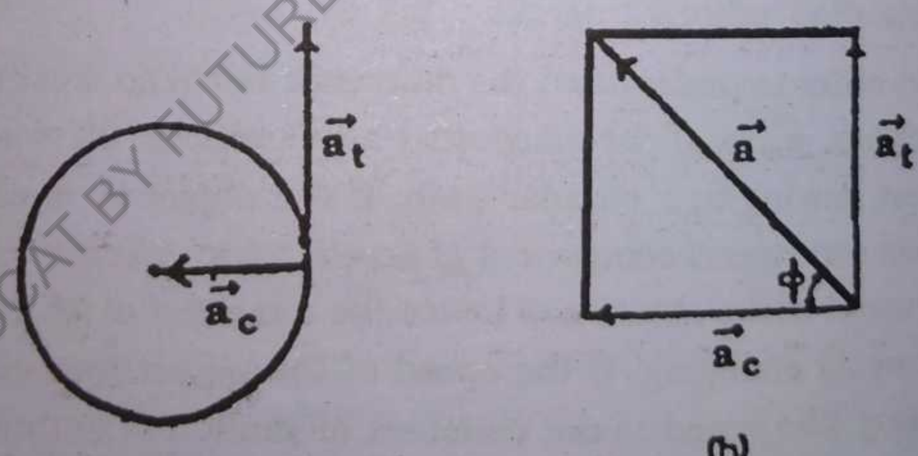


Fig. 4.13 (a)

Alternatively, Fig 4.14 shows the three vectors $\vec{r}$, $\vec{v}$ and $\vec{a}$ representing position vector, velocity vector and centripetal acceleration vector respectively for the same instant, all drawn from the centre of the circle. The velocity vector $\vec{v}$, is always perpendicular

to the position vector, $\vec{r}$, for a circular path, since this is a unique geometrical property of a circle as shown in Fig 4.13. This is true whether the speed is constant or not. In other words it can be stated that the velocity vector, $\vec{v}$, leads the position vector, $\vec{r}$, by 90°.

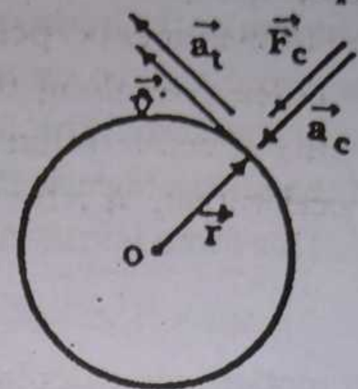


Fig: 4.14

The direction of the centripetal acceleration vector is perpendicular to the velocity vector, $\vec{v}$, that is, the acceleration vector $\vec{a}$, leads the velocity vector, $\vec{v}$, by 90°. Thus the centripetal acceleration vector, $\vec{a}_c$, leads the position vector, $\vec{r}$, by 180°, exactly in an opposite direction of the position vector as drawn in Fig 4.14. Because the position vector, $\vec{r}$, is directed away from the centre of the circular path, therefore, the centripetal acceleration vector, $\vec{a}_c$, is always directed toward the centre of the circulation path. The magnitude of these vectors are constant in time; but the vectors themselves are certainly not, only their directions are constantly changing

4.15 CENTRIPETAL FORCE

Consider a ball of mass 'm' tied to a string of length r is being whirled with a constant speed in a circular orbit as shown in Fig 4.15. We know the velocity vector, $\vec{v}$, changes its direction

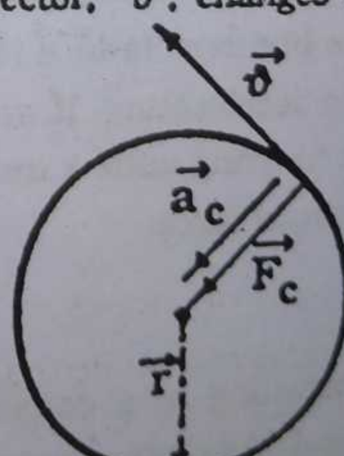


Fig. 4.15 A ball of mass m rotating in a circular orbit.

continuously during the circular motion, the ball experience a centripetal acceleration which is directed toward the centre of the orbit.

According to first law of motion the inertia of ball tends to maintain motion in straight line path; however, the string does not allow this to happen by exerting a force on the ball such that the ball follow its circular path. This force (the force of tension) is directed along the length of the string toward the centre of the circle as shown in the Fig (4.13). This force is called centripetal force and represented by $\vec{F}_c$.

Using second law of motion, we calculate the magnitude, F_c, of centripetal force $\vec{F}_c$.

$$F_c = m\,a_c$$

substituting $a_c = \frac{\vartheta^2}{r}$

$$F_c = \frac{m\vartheta^2}{r}$$

$$F_c = \frac{mr^2\omega^2}{r} = mr\omega^2$$

The centripetal force, vector, $\vec{F}_c$ acts toward the centre of circular path along which the object moves. In the absence of such a force, the object will no longer move in its circular path; instead it would move along a straight line path tangent to the circle.

Some readers may be familiar with the term centrifugal force or centrifugal acceleration, such a force or acceleration only occurs when the observer is in a rotating frame of reference, that is, the observer is accelerating. If we restrict our discussion to observer "at rest" or "moving with a uniform velocity", we shall never encounter centrifugal force.

Example 4.5

A car traveling at a constant speed of 72 km/h rounds a curve of radius 100 m. What is its acceleration?

Solution

The magnitude of the centripetal acceleration is given by Eq

$$a_c = \frac{\vartheta^2}{r}$$

$$\vartheta = \frac{72000\text{m}}{3600\text{s}} = 20\text{ms}^{-1}$$

$$a_c = \frac{\vartheta^2}{r} = (20\text{ms}^{-1})^2 / 100\text{m}$$

$$= 4.0\text{m.s}^{-2}$$

The direction of $\vec{a_c}$ at each instant is perpendicular to the velocity vector and directed toward the centre of the circle.

Example 4.6

A 200 gram ball is tied to the end of a cord and whirled in a horizontal circle of radius 0.6 m. If the ball makes five complete revolutions in 2s, determine the ball's linear speed, its centripetal acceleration, and the centripetal force.

Solution

The ball makes five revolutions in 2s, traveling a distance of $2\pi r$ in each revolution

Time for one revolution $T = \frac{2}{5}\text{s} = 0.4\text{s}$

The linear speed of the ball is

$$\vartheta = \frac{2\pi r}{T}$$

$$\vartheta = \frac{2\pi \times 0.6\text{m}}{0.4\text{ s}} = 9.42\text{ms}^{-1}$$

The centripetal acceleration is

$$a_c = \frac{\vartheta^2}{r} = (9.42\text{ms}^{-1})^2 / 0.6\text{m} = 148\text{ms}^{-2}$$

The centripetal force is

$$F_c = ma_c = \left(\frac{200}{1000}\,kg\right)(148\ ms^{-2}) = 29.6N$$

Example 4.7

Calculate the centripetal acceleration and centripetal force on a man whose mass is 80 kg when resting on the ground at the equator if the radius of earth R is 6.4×10^6 m.

Solution

Due to rotation of the earth, the man at equator moves in a circle whose radius is equal to the radius of the earth. The man makes one rotation in about 24h; hence, his speed is given by

$$\vartheta = \frac{2\pi r}{T} = \frac{2\pi\,(6.4 \times 10^6 m)}{24\,(60)\,(60)s} = 465\ ms^{-1}$$

The centripetal acceleration is

$$a_c = \frac{\vartheta^2}{R} = \frac{(465ms^{-1})^2}{6.4 \times 10^6 m} = 3.37 \times 10^{-2}\ ms^{-2}$$

The centripetal force is

$$F_c = ma_c = (80\ kg)\,(3.37 \times 10^{-2}\ ms^{-2})$$

$$F_c = 2.69\ N.$$

4.16 Some important relations of linear motion and Angular motion.

When an object is constrained to rotate about an axis fixed in space, the angular variables θ, ω and α are related to each other in exactly the same way as are the variables, s, ϑ and a for motion along a straight line, as shown in Table 4.1.

TABLE 4.1

Equations for constant angular acceleration, $\vec{\alpha}$, along the axis of rotation and their translational motion analogs. In using

these equations, one direction along the rotation axis is taken as positive and the other as negative. θ, ω and α can be positive or negative.

Table 4.1

Linear motion	**Rotational Motion**
Constant linear acceleration, a.	Constant angular acceleration, α.
$s = \vartheta t$	$\theta = \omega t$
$\vartheta_f = \vartheta_i + at$	$\omega_f = \omega_i + \alpha t$
$\vartheta_{av} = \dfrac{\vartheta_f + \vartheta_i}{2}$	$\omega_{av} = \dfrac{\omega_f + \omega_i}{2}$
$s = \vartheta_i t + \frac{1}{2} at^2$	$\theta = \omega_i t + \frac{1}{2} \alpha t^2$
$\vartheta_f^2 - \vartheta_i^2 = 2as$	$\omega_f^2 - \omega_i^2 = 2\alpha\theta$

Problems

1. A rescue helicopter drops a package of emergency ration to a stranded party on the ground. If the helicopter is traveling horizontally at 40 m/s at a height of 100 m above the ground. (a) where does the package strike the ground relative to the point at which it was released? (b) What are the horizontal and vertical component of the velocity of the package just before it hits the ground ?

(Ans: (a) 180 m (b) 40 m/s, –44.1 m/s)

2. A long-jumper leaves the ground at an angle of 20° to the horizontal and at a speed of 11 m/s (a) How far does he jump ? What is the maximum height reached ? Assume the motion of the long jumper is that of projectile.

(Ans: (a) 7.94 m (b) 0.722 m)

3. A stone is thrown upward from the top of a building at an angle of 30 to the horizontal and with a intial speed of 20 m/s. If the height of building is 45 m.(a) Calculate the total time the stone in flight (b) What is the speed of stone just before it strikes the ground? (c) Where does the stone strike the ground ?

(Ans: (a) t = 4.22 s (b) V = 35.8 m/s

(c) 73.0 m from the base of building.)

4. A ball is thrown in horizontal direction from a height of 10 m with a velocity of 21 m/s (a) How far will it hit the ground from its initial position on the ground? and with what velocity ?

(Ans: [30 m, 25.2 m/s])

5. A rocket is launched at an angle of 53° to the horizontal with an initial speed of 100 m/s. It moves along its initial line of motion with an acceleration of 30 m/s^2 for 3s. At this time the engine fails and the rocket proceeds to move as a free body. Find (a) the maximum altitude reached by the rocket (b) its total time of flight, and (c) its horizontal range.

(Ans: (a) 1.52×10^3 m (b) 36.1 s (c) 4.05 km.)

6. A diver leaps from a tower with an initial horizontal velocity component of 7 m/s and upward velocity component of 3 m/s. Find the component of her position and velocity after 1 second

(Ans: V_x = 7 m/s, V_y = -6.8 m/s)

7. A boy standing 10 m from a building can just barely reach the roof 12 m above him when he throws a ball at

8.

9.

10

11

the optimum angle with respect to the ground. Find the initial velocity component of the ball.

(Ans: $V_{ox} = 6.41$ m/s, $V_{oy} = 15.3$ m/s)

8. A marter shell is fired at a ground level target 500 m distance with an initial velocity of 90 m/s. What is its launch angle ?

(Ans. 71.4°)

9. What is the take off speed of a locust if its launch angle is 55° and its range is 0.8m ?

(Ans: 2.9 m/s)

10. A car is travelling on a flat circular track of radius 200 m at 20 m s^{-1} and has a centripetal acceleration $a_c = 4.5$ m s^{-2} (a) If the mass of the car is 1000 kg, what frictional force is required to provide the acceleration? (b) If the coefficient of static friction μ_s is 0.8, what is the maximum speed at which the car can circle the track?

(Ans: (a) 4500 N, (b) 39.6 m/s)

11. The turntable of a record player rotates initially at a rate of 33 rev/min and takes 20 s to come to rest (a) What is the angular acceleration of the turntable, assuming the acceleration is constant? (b) How many rotation does the turntable make before coming to rest? (c) If the radius of the turntable is 0.14 m, what is the initial linear speed of a bug riding on the rim? (d) What is the magnitude of the tangential acceleration of the bug at time t = 0 ?

(Ans: (a) - 0.173 rad/s^2 (b) 5.5 rev
(c) 0.484 m/s (d) 0.0242 m/s^2)

12. Tarzan swings on a vine of length 4m in a vertical circle under the influence of gravity. When the vine makes an angle of $\theta=20°$ with the vertical, Tarzan has a speed of 5 $m\ s^{-1}$. Find (a) his centripetal acceleration at this instant, (b) his tangential acceleration, and (c) the resultant acceleration.

(Ans: (a) 6.25 m/s^2 (b) 3.35 m/s^2 (c) 7.09 m/s^2)

Torque Angular Momentum and Equilibrium

In earlier chapters, we have discussed the linear and angular motions of bodies in detail. In our daily life, we come across various types of motion, for example the motion of a train from one station to an other, the motion of a car along a road, the motion of a ceiling fan, the whirling of a stone which is tied to one end of a string, the other end being held in the hand, etc.The above motion can be divided into two groups (i) translatory motion and (ii) rotatory motion.

The motion of the train and the car belongs to the first group (translatory motion) whereas the motion of the fan and the stone belongs to the second group (rotatory motion).

Even a common man can differentiate these two types of motion according to his ability. As a student of Physics one can difine these motion as follows:

Translatory motion:- Consider a frame of reference (x`, y` z`) which is imagined to be rigidly fixed to an object. For an observer the motion of the object is said to be purely translatory if the axes of the frame of reference of the object remains always parallel to the corresponding axes of observer's frame of reference (x, y, z). In a translatory motion the object may not be necessarily moving along a straight line. Fig.5.1, shows translational motion of an object moving from A to B and C. Observe that throughout the motion every point of the object undergoes same displacement as every other point. We can assume the body to be a particle because in

describing the motion of one point on the body we have described the motion as a whole.

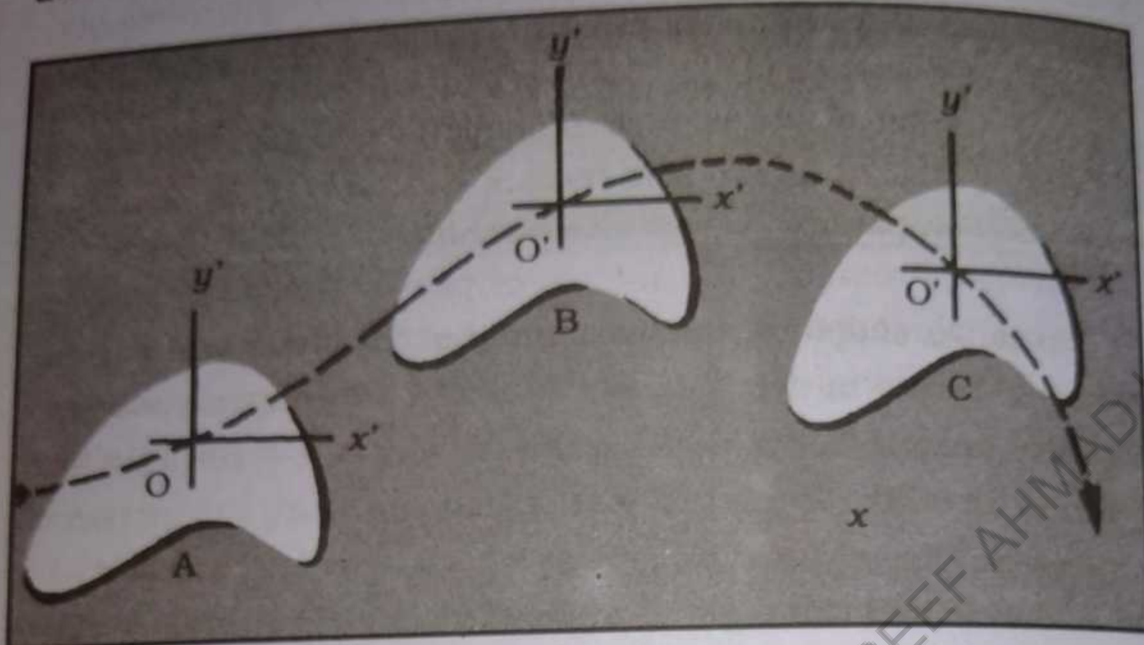


Fig. 5.1

Rotatory motion:- A body (a rigid body) is said to possess a purely rotatory motion if every constituent particle of the body moves in a circle, the centres of which are on a straight line called the axis of rotation.This motion is of two kinds (i) spin motion and (ii) orbital motion.

Before we define and give some examples of spin and orbital motions,we first define the axis of rotation.

By the axis of rotation, we mean the line about which a body rotates. If the line (the axis of rotation) passes through the body itself the corresponding motion is called the spin motion. Every point of the spinning object moves along an arc of a circle in a small interval of time and the centres of all these circles lie along a straight line. This straight line is called the axis of rotation. However, if the axis of rotation does not pass through the body, we call such a rotatory motion as the orbital motion.

Examples of spin and orbital motions:

The daily rotation of the earth about its own axis is an exam-

ple of spin motion. Rotation of fly wheel about its axle is another example of spin motion. A rotating top presents a familiar example of spin motion. If you happen to visit a non-electrified village, you would see that sugar-cane crushing machine is run by a camel or by a bullock that moves in a circular path around the machine. This motion of the camel or the bullock is an orbital motion. The axis of rotation passes through the machine and is perpendicular to the plane of rotation. The motion of planets round the sun is also an orbital motion. In this example the axis of rotation passes through the centre of the sun. The motion of electrons round the nucleus is an example of orbital motion. Here the axis of rotation passes through the centre of the atom.

5.1 TORQUE

Consider a particle of mass 'm' which is acted upon by a force $\vec{F}$. Let $\vec{r}$ be the position vector of the particle. This is also the position vector of the point of application of the force. We can resolve this force into two rectangular components (i) $\vec{F}_{\parallel}$ and (ii) $\vec{F}_{\perp}$. The component $\vec{F}_{\parallel}$ acts in the direction of $\vec{r}$ whereas $\vec{F}_{\perp}$ acts in a direction perpendicular to $\vec{r}$ as shown in the Fig.5.2.

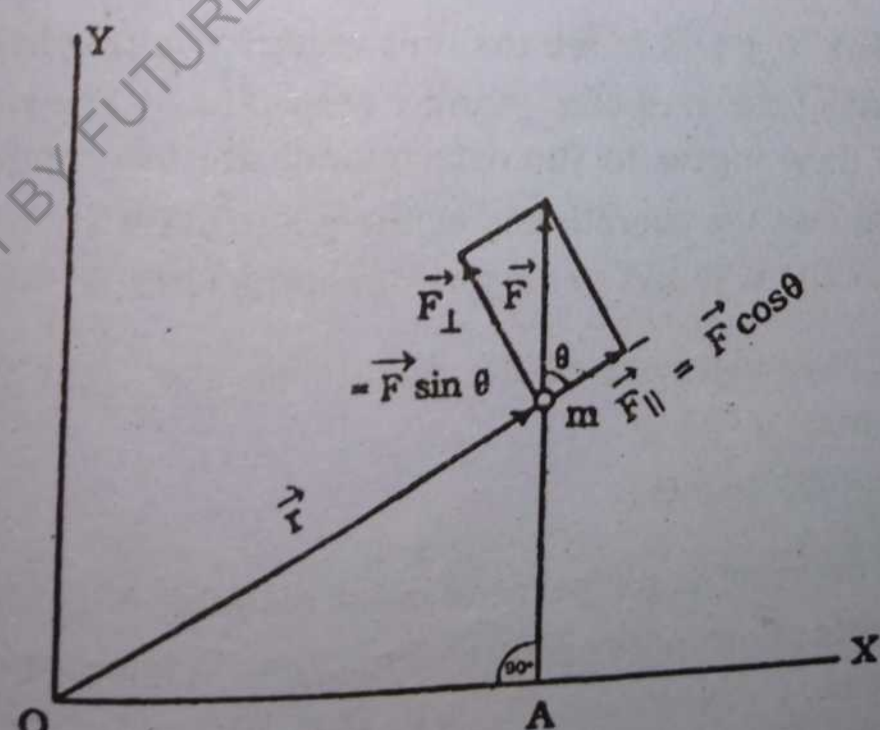


Fig. 5.2

The force $\vec{F}_{\parallel}$ can pull the mass but cannot rotate it. It is the force $\vec{F}_{\perp}$ which produces rotation. Let r and $F_{\perp}$ be the magnitudes of $\vec{r}$ and $\vec{F}_{\perp}$ respectively. The magnitude of the torque vector $\vec{\tau}$ produced by the force $\vec{F}$ about the centre 'O' is defined as

$$\tau = r F_{\perp} = r\ F\sin\theta \qquad (5.1)$$

Where θ is the smaller angle between the positive directions of $\vec{r}$ and $\vec{F}$, i.e. $\theta \leqq 2\pi - \theta$

Using vector notation, Eq. 5.1 can be re-written as

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (5.2)$$

The torque vector $\vec{\tau}$ which is the vector product of $\vec{r}$ and $\vec{F}$ is directed along the normal to the plane defined by $\vec{r}$ and $\vec{F}$. The direction of $\vec{\tau}$ can also be given by the right hand rule. <u>Right-hand rule</u>:- Point your fingers of right hand towards $\vec{r}$ and curl them from $\vec{r}$ to $\vec{F}$. Then the direction of the thumb will give the direction of $\vec{r} \times \vec{F}$.

We can represent the torque vector $\vec{\tau}$ in the determinant form as shown below :

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \qquad (5.3)$$

Here i, j and k are the unit vectors pointing in the positive directions of the axes of x, y and z respectively. The elements x, y and z as they appear in the determinant are the components of $\vec{r}$. They are also the coordinates of the point mass 'm' whereas F_x, F_y and F_z are the x, y, and z components of the force $\vec{F}$ respectively.

A rearrangement of the factors on the right-hand side of Eq.5.1 gives

$$\tau = (F)\ (r\sin\theta)$$

Here '$r\sin\theta$' is the perpendicular distance of the line of action of the force $\vec{F}$ from the centre of rotation. It is represented by the line segment OA in the Fig.5.1. This perpendicular distance is called the moment arm of the force. Thus the magnitude of the

torque is the ordinary product of the magnitude of the force and its moment arm. i.e.

Magnitude of torque = (magnitude of force) × (moment arm)

This means the greater the force, the larger will be the torque and larger the moment arm the greater will be the torque. It is, therefore, easier to rotate a body by applying a given force when its line of action is at a greater distance (moment arm) from the centre of the rotation than when it acts closer to the body. In the extreme case when the line of action passes through the centre of rotation, the body stops to rotate because the moment arm is zero. Thus a tangential push on the rim of a wheel will cause a rotation, while a similar push along the axle of the wheel is unable to produce any such rotation. In Fig 5.3 a wheel rotates about an axis passing

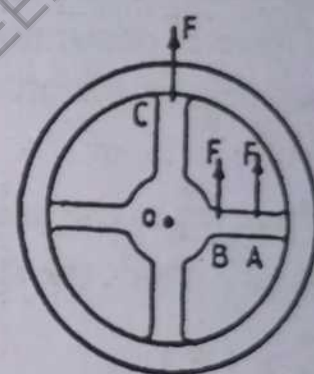


Fig. 5.3 The Turning effect of a force is greater, the farther is the line of action of the force from the axis of rotation.

through its centre 'O'. The turning effect on the wheel has been shown by applying a force at different points on the wheel.

A body can rotate clockwise or counter-clockwise. As a convention, counter-clockwise rotation is taken as positive while a clockwise rotation as negative. Hence a torque which produces a counter-clockwise rotation is considered to be positive and that producing a clockwise rotation is taken as negative.

Two forces which are equal in magnitude but opposite in direction and not acting along the same line constitute a couple as shown in Fig.5.4. The forces constituting the couple are represented by $\vec{F}$ and $-\vec{F}$.

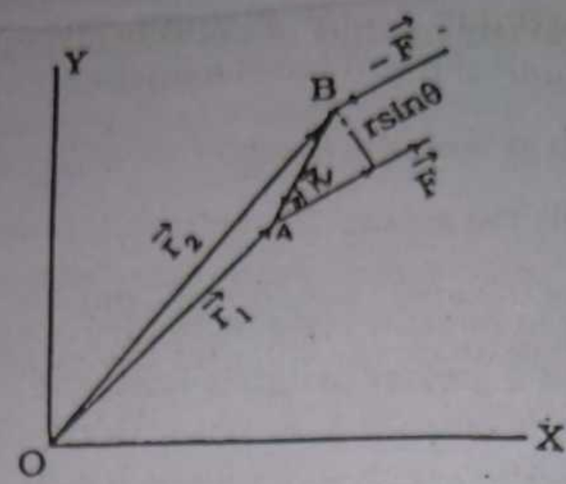


Fig: 5.4 Two forces equal in magnitude but opposite in direction and not acting along the same line constitute a couple.

Consider a couple composed of two forces $\vec{F}$ and $-\vec{F}$ acting at the points 'A' and 'B' respectively as shown in Fig. 5.4. The moment of force $\vec{F}$ about the origin '0' is

$$\vec{r}_1 \times \vec{F}$$

where $\vec{r}_1$ is the position vector of the point A. The moment of the force $-\vec{F}$ about the same point is

$$\vec{r}_2 \times (-\vec{F})$$

Where $\vec{r}_2$ is the position vector of the point B. The total moment of the two forces is given as

$$\vec{\tau} = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F})$$

$$= \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$

$$= \vec{r} \times \vec{F} \qquad 5.4$$

Where $\vec{r}$ is the displacement vector from B to A. The vector $\vec{\tau}$ is called the moment of the couple ($\vec{F}$ - $\vec{F}$). Since $\vec{\tau}$ is a vector product of two vectors $\vec{r}$ and $\vec{F}$, its direction is along the normal

to the plane containing $\vec{r}$ and $\vec{F}$ and is given by the right-hand rule. The magnitude of the vector $\vec{\tau}$ by definition

$$\tau = (r)(F)\sin(180^\circ - \theta)$$

$$\tau = r F \sin\theta \qquad (5.5)$$

Where θ is the angle between $\vec{r}$ and $-\vec{F}$. It is evident from Fig 5.4 that $r\sin\theta$ is the perpendicular distance between the lines of action of the forces $\vec{F}$ and $-\vec{F}$. Let us denote this distance by 'd' ($d = r\sin\theta$). The Eq.5.5 takes the form.

$$\tau = (F)(d) = Fd \qquad (5.6)$$

The perpendicular distance 'd' is called the moment arm of the couple. Consequently.

Magnitude of the moment of a couple =

Magnitude of any of the forces forming the couple x

Area of the couple.

As $\vec{r}$ is the displacement vector from $-\vec{F}$ to $\vec{F}$, it is independent of the location of origin. Hence the moment of a given couple is independent of the location of origin.

5.2 CENTRE OF MASS

In translational motion each point on a body undergoes the same displacement as any other point as time goes on, so that motion of one particle represents the motion of the whole body. But even when a body rotates or vibrates as it moves, there is one point on the body, called the centre of mass, that moves in the same way that a single particle would move under the influence of the same external forces.

The centre of mass of a body or a system of particles is defined to be a point which moves as if the total mass of the body or the system of particles were concentrated there and all the applied forces were acting at that point. This means that we can describe the motion of the whole system or the body by the motion of

their centres of mass. For example, consider a rectangular block of wood lying on a smooth horizontal surface. Let the block be acted upon by a number of forces. In order to describe the motion of the block as a whole we assume that these forces were acting at the centre of mass which is the geometrical centre of the block and where the total mass is supposed to be concentrated. We then, find the resultant of these forces and apply Newton's second law of motion to determine the acceleration and hence the velocity of the centre of mass at any instant of time by using initial condition of motion. The motion of the block is same as the motion of the centre of mass.

The centre of mass is often confused with the centre of gravity. The two terms are so similar in many respects that one can use the two terms interchangeably. The centre of gravity of an extended object coincides with its centre of mass if the object is in a completely uniform gravitational field. If this is not the case the centre of gravity does not coincide with the centre of mass.

Let x_c, y_c and z_c be the coordinates of the centre of mass, then in a completely uniform gravitational field, these are given by

$$x_c = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$$y_c = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

$$z_c = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

Where x_i, y_i, z_i are the coordinates of the particle of mass m_i.

Example 5.1

Locate the centre of mass of four particles which are placed at the four corners of a square of sides 2m each as shown in Fig.5.5.

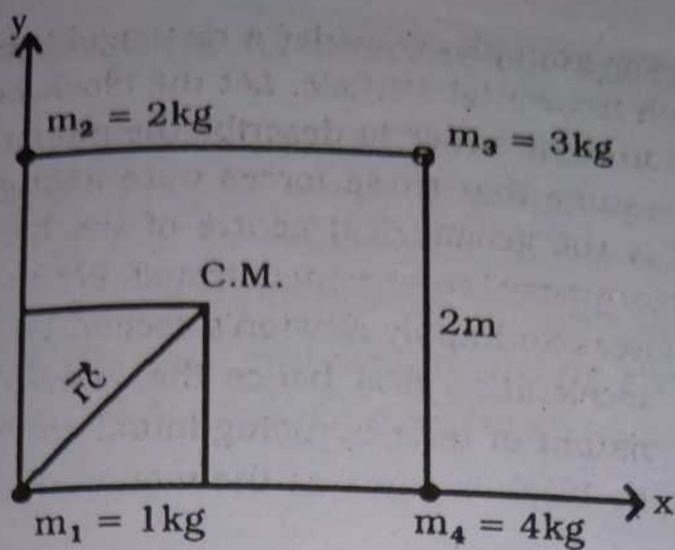


Fig. 5.5

Solution: This is a two dimentional problem. For the sake of simplicity we take the two adjacent sides of the square in the direction of the coordinate axes. Let x_c and y_c be the x and y coordinates of the centre of mass then

$$x_c = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{1 \times 0 + 2 \times 0 + 3 \times 2 + 4 \times 2}{1 + 2 + 3 + 4}$$

$$= \frac{14}{10} = 1.4$$

$$y_c = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{1 \times 0 + 2 \times 2 + 3 \times 2 + 4 \times 0}{1 + 2 + 3 + 4}$$

$$= \frac{10}{10} = 1$$

Hence the coordinates of the centre of mass are the point (1.4, 1.0)

5.3 EQUILIBRIUM AND CONDITIONS OF EQUILIBRIUM

A body is said to be in equilibrium if it is at rest or is moving with a uniform velocity. A body at rest is said to be in static equilibrium, while a body in uniform motion along a straight line is said to be in dynamic equilibrium. In both the cases the body do not possess any acceleration neither linear nor angular. Hence all the bodies in equilibrium do not possess any acceleration.

(A) Static equilibrium

A book lying on a horizontal table is in static equilibrium. Building and bridges are also in static equilibrium. There are many more examples of static equilibrium around us.

(B) Dynamic equilibrium

Consider a vertically downward motion of a small steel ball through a viscous liquid like mustard oil contained in a vertical tube which is clamped with a stand. The ball is dropped gently into the oil. Initially the ball has accelerated motion. But after covering a certain distance it attains a uniform vertical velocity. This uniform vertical motion of the steel ball is an example of dynamic equilibrium. In this example the force of gravity acting vertically downward on the ball is counter balanced by the viscous force acting vertically upward on it. Hence a net external force acting on the ball is zero. As a result of this the ball acquires uniform vertical motion.

Another example of dynamic equilibrium can be presented by considering the jumping of a paratrooper from an helicopter. In this case the force of gravity acts on the paratrooper in a vertically downward direction whereas the reaction of air on the parachute acts in the vertically upward direction. These two forces balance each other as shown in Fig. 5.6. Thus the net force acting on

the system (the parachute and the paratrooper) is zero. Hence the system moves downward with a uniform velocity.

Fig. 5.6 After a parachute opens and falls a certain distance, it moves downward thereafter with a uniform velocity.

5.4 FIRST CONDITION OF EQUILIBRIUM

The condition states that a body will be in equilibrium if the resultant of all the forces acting on it is zero. This condition is referred to as the first condition of equilibrium.

Let $\vec{F_1}, \vec{F_2}, \ldots\ldots, \vec{F_n}$ be the n external forces acting on a body. The first condition of equilibrium states that

$$\vec{F_1} + \vec{F_2} + \ldots\ldots + \vec{F_n} = 0 \qquad 5.7\ (a)$$

Using the summation sign, the above Eq 5.7 can be written as

$$\sum_{i=1}^{n} \vec{F_i} = 0 \qquad 5.7(b)$$

If we restrict to forces in one plane, say, the x-y plane, then we write

$$\vec{F_1} = F_{1x} i + F_{1y} j$$

Where F_{1x} is x - component of the force $\vec{F_1}$;

F_{1y} is y - component of the force $\vec{F_1}$;

and i , j are the unit vectors in the direction of x and y axes respectively. Thus the Eq. 5.7 can be written as

$$(F_{1x} i + F_{1y} j) + (F_{2x} i + F_{2y} j) + \cdots\cdots\cdots + (F_{nx} i + F_{ny} j) = 0$$

or

$$(F_{1x} + F_{2x} + \ldots\ldots + F_{nx}) i + (F_{1y} + F_{2y} + \ldots\ldots + F_{ny}) j = 0$$

Let $\vec{F}$ be the resultant of the forces $\vec{F_1}, \vec{F_2}, \ldots\ldots, \vec{F_n}$

$$\therefore \quad \vec{F} = \vec{F_1} + \vec{F_2} + \ldots\ldots\ldots, \vec{F_n}$$

or

$$F_x i + F_y j = (F_{1x} + F_{2x} + \ldots\ldots + F_{nx}) i + (F_{1y} + F_{2y} + \ldots\ldots + F_{ny}) j$$

Equating the x and y components of the forces on both the sides of the above equation we get

$$F_x = F_{1x} + F_{2x} + \ldots\ldots + F_{nx}$$

and

$$F_y = F_{1y} + F_{2y} + \ldots\ldots + F_{ny}$$

Since $\vec{F_1} + \vec{F_2} + \ldots\ldots + \vec{F_n} = 0$

So $\vec{F} = 0$.

Since $\vec{F}$ is zero.

$$F_x = 0 \text{ and } F_y = 0$$

or $F_{1x} + F_{2x} + \ldots\ldots + F_{nx} = 0$. $F_{1y} + F_{2y} + \ldots\ldots + F_{ny} = 0$

Using the summation convention

$$\left.\begin{aligned} &\sum_{i=1}^{n} F_{ix} = 0 \\ \text{and} \\ &\sum_{i=1}^{n} F_{iy} = 0 \end{aligned}\right\} \qquad 5.8(a)$$

For the sake of simplicity and convenience we omit the index 'i' from the summation sign in the above equation and write the condition of equilibrium in the form

$$\left.\begin{aligned} &\Sigma F_x = 0 \\ \text{and} \\ &\Sigma F_y = 0 \end{aligned}\right\} \qquad 5.8(b)$$

One important point to be noted here is that all the component forces should be written with their proper sign.

Let $\theta_1, \theta_2, \ldots, \theta n$ be the angles which the forces $\vec{F_1}, \vec{F_2}, \ldots \vec{F_n}$ make with x-axis respectively as shown in Fig. 5.7

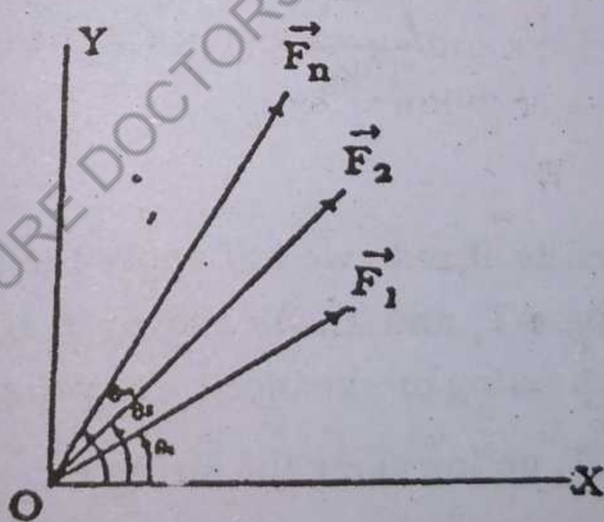


Fig: 5.7 Vectors $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_n}$ are drawn in x, y - Plane

Then we have

$F_{1x} = F_1 \cos\theta_1$, $F_{2}x = F_2 \cos\theta_2, \ldots\ldots F_{nx} = F_n \cos\theta_n$

and

$F_{1y} = F_1 \sin\theta_1$, $F_{2y} = F_2 \sin\theta_2, \ldots\ldots F_{ny} = F_n \sin\theta_n$

The first condition of equilibrium is then written as

$$\left.\begin{aligned} \Sigma F_x &= \sum_{i=1}^{n} F_{ix} = \sum_{j=1}^{n} F_j \cos\theta_j = 0 \\ \text{and} \quad \Sigma F_y &= \sum_{i=1}^{n} F_{iy} = \sum_{j=1}^{n} F_j \sin\theta_j = 0 \end{aligned}\right\} \qquad 5.8(c)$$

Example 5.2:

A block of mass 20 kg is suspended as shown in Fig. 5.8. Find the tensions T_1 and T_2 in the two strings. Take the value of $g = 9.8\ m/s^2$.

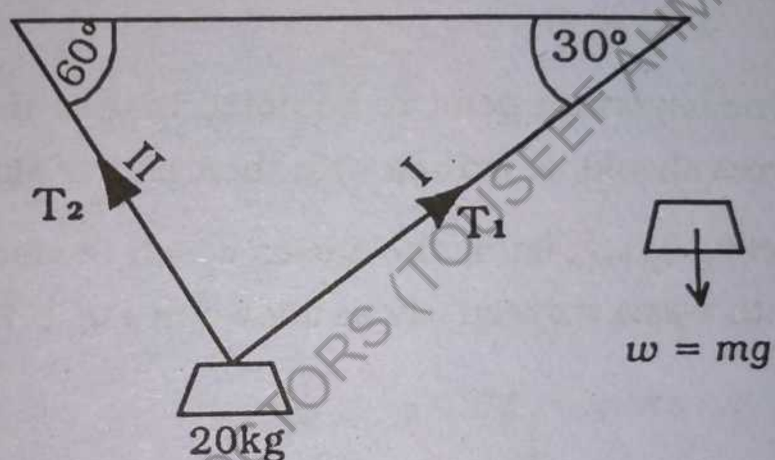


Fig: 5.8

Solution:

Since the block is at rest, we will apply the first condition of equilibrium to evaluate T_1 and T_2. As shown in the force diagram there are three forces acting on the block. They are

(i) the tension T_1 acting along the string I.

(ii) the tension T_2 acting along the string II

(iii) the weight W of the block acting vertically downward.

We resolve these forces into their respective x and y components. Applying first condition of equilibrium for the x-components of the forces we have

$$\Sigma F_x = 0$$

Since the weight of the block 'W' has no component along the horizontal direction, the above summation reduces to

$$T_1 \cos 30° - T_2 \cos 60° = 0$$

$$T_1 \times \frac{\sqrt{3}}{2} - T_2 \times \frac{1}{2} = 0$$

$$T_2 = \sqrt{3}\ T_1$$

The first condition of equilibrium for the y-components of the forces i.e.

$$\Sigma F_y = 0$$

$\therefore$

$$T_1 \sin 30° + T_2 \sin 60° - w = 0$$

$$T_1 \times \frac{1}{2} + T_2 \times \frac{\sqrt{3}}{2} - w = 0$$

$$T_1 + \sqrt{3}\ T_2 = 2w$$

$$T_1 + \sqrt{3} \times \sqrt{3}\ T_1 = 2w$$

$$4T_1 = 2w$$

$$T_1 = \frac{w}{2} = \frac{20(kg) \times 9.8(m/s^2)}{2} = 98N$$

solving for T_2, we get

$$T_2 = 98\sqrt{3}\,N = 169.74N$$

5.5 SECOND CONDITION OF EQUILIBRIUM

Earlier we have studied that whenever, a net torque acts on a body, rotation is produced in it.The body is said to in rotational equilibrium, if the vector sum of the torques acting on it is zero. Then if $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3, ...\vec{\tau}_n$ are the torques on the body, then

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 ...... + \vec{\tau}_n = 0$$

$$\sum_{i=1}^{n} \vec{\tau}_i = 0$$

Where $\vec{\tau}_i$ is the moment of the ith force. For convenience we omit the subscript from the summation sign. Thus we have

$$\Sigma\vec{\tau}_i = 0 \qquad 5.9(a)$$

Thus the two conditions of equilibrium written together are

(i) $\Sigma\vec{F} = 0, \Sigma\vec{F}_x = 0, \Sigma\vec{F_y} = 0$

(ii) $\Sigma\vec{\tau} = 0$, about any axis (5.9 (b)

Example 5.3

A 15m ladder weighing 600N rests against a smooth wall at a point 12m above the ground. The centre of gravity of the ladder is one third the way up. A man weighing 400N climbs half way up the ladder. Assuming that the wall is smooth, find the reaction of the ground and the wall.

Solution:

By the term smooth we mean that there is no friction and that the reaction of the wall on the ladder is perpendicular to the wall. Here we isolate the ladder as the body under consideration. The forces acting on the ladder are shown in Fig.5.9 (b). The weight W of the ladder acts vertically downward at the point C so that m $\overline{AC} = \frac{1}{3}$ m$\overline{AB}$. The weight W of the man acts vertically downward at the mid point D of the ladder and $\vec{F}$ the reaction force of the wall acts perpendicular to the wall. The unknown force of reaction of the ground on the ladder is represented in terms of its horizontal and vertical component H and V respectively. From the geometry of the Fig. 5.9. we get

$$m\overline{AE} = 9m, \; m\overline{AK} = 3m \text{ and } m\overline{AJ} = 4.5m$$

Applying the first condition of equilibrium to the ladder we get

$$\Sigma F_x = 0$$

$$\therefore \quad H - F = 0 \qquad 5.10(a)$$

$$H = F$$

$$\Sigma F_y = 0 \qquad 5.10(b)$$

$\therefore \quad V - w - W = 0$

$V - 600 - 400 = 0$

$V = 1000\ N$

To write the torque equation we choose an axis passing through the point A where the unknown force (with unknown direction) of reaction acts. By doing so the unknown force will not appear in this equation.

Now taking the moment about the point A we get

$$\Sigma\tau = 0$$

$$F \times (m\overline{AG}) - w \times (m\overline{AK}) - w \times (m\overline{AJ}) = 0$$

or $$F \times 12 - 600 \times 3 - 400 \times 4.5 = 0$$

$$12\,F - 1800 - 1800 = 0$$

$$12\,F = 3600$$

$$F = 300N$$

Since $\quad H = F$

$\therefore \quad H = 300\ N$

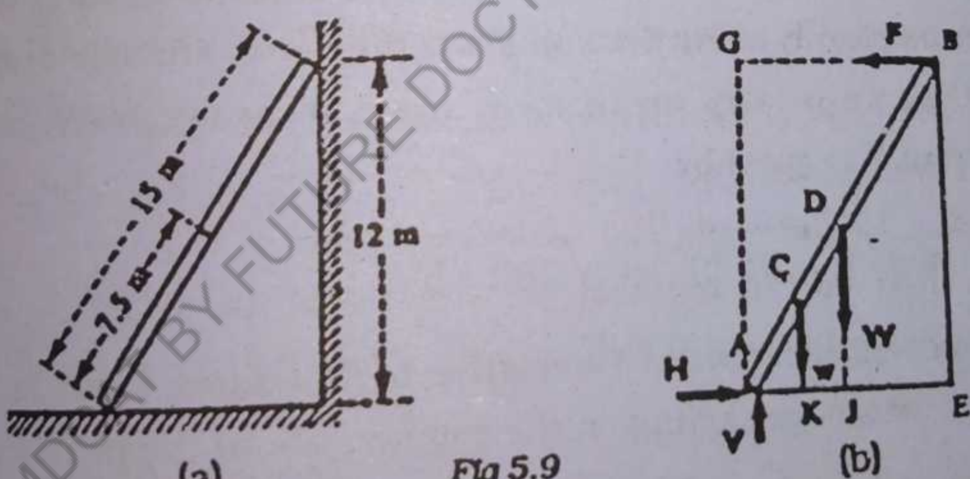


Fig 5.9

The magnitude of the force of reaction of the ground is R.

$$R = \sqrt{H^2 + V^2}$$

$$= \sqrt{(300)^2 + (1000)^2}$$

$$= 1044\ N$$

The direction of the reaction force may be found by evaluating the angle which it makes with the horizontal. Let θ be such an angle. as shown in Fig. 5.9 (a)

$$\tan\theta = \frac{V}{H} = \frac{1000}{300} = \frac{10}{3} \approx 3.33$$

$$\theta = 73.3^{\circ}$$

Example 5.4

A cylinder of weight $\vec{W}$ and radius R is to be raised on to a step of height 'h' as shown in Fig. 5.10. A rope is wrapped around the cylinder and pulled horizontally. Assuming that the cylinder does not slip on the step, find the minimum force $\vec{F}$ necessary to raise the cylinder and the reaction force at the point P.

Solution:

When the cylinder is just ready to be raised, the reaction force at Q goes to zero. Hence only three forces are acting on the cylinder as shown in Fig.5.10(b). From the dotted triangle drawn in Fig.5.10(a), we see that the moment arm d of the weight $\vec{W}$ relative to the point P is given by

$$d = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

The moment arm of $\vec{F}$ relative to the same point is 2R-h. Hence the net torque acting on the cylinder about the point P is

$$Wd - F\,(2R-h).$$

Since the system is at rest, we apply the second condition of equilibrium to get

$$\Sigma\,\tau = 0$$

$$\text{or } Wd \quad F\,(2R-h) = 0$$

$$F = \frac{Wd}{2R - h} = \frac{W\sqrt{2Rh - h^2}}{2R - h}$$

$$= W\sqrt{\frac{h}{2R - h}}$$

Thus the second condition of equilibrium is sufficient to obtain the magnitude of $\vec{F}$. We can determine the reaction force $\vec{N}$ at P by using the first condition of equilibrium. Thus we have

$$\Sigma F_x = 0$$

$$\text{or } F - N\cos\theta = 0$$

$$F = N\cos\theta$$

$$\text{and } \Sigma F_y = 0$$

$$N\sin\theta - W = 0$$

$$N\sin\theta = W$$

From these equations we can easily get

(i) $\tan\theta = \dfrac{W}{F}$ and (ii) $N = |\vec{N}| = \sqrt{F^2 + W^2}$

As an example if we take W = 500N, h = 0.3m and R = 0.8m, then

$F = 385$ N, $\theta = 52.4^\circ$ and $|\vec{N}| = 631$ N

This problem can also be solved by noting that since only three forces act on the body, they must be concurrent. Hence the reaction force $\vec{N}$ must pass through the point C where $\vec{F}$ and $\vec{W}$ intersect. The three forces from the sides of the triangle as shown in fig. 5.10(c). Expressions (i) and (ii) are obtained directly from this.

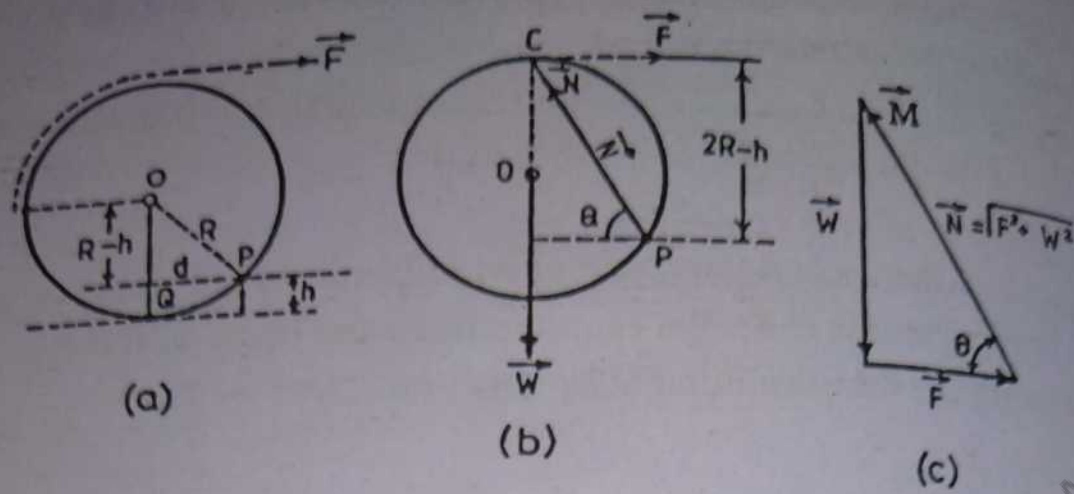


Fig. 5.10. (a) A cylinder of weight W pulled by a force F over a step.
(b) The free body diagram of the cylinder when it is just ready to be raised.
(c) The vector sum of the three forces is zero.

5.6 LOCATION OF AXIS

We will now show that if a body is in translational equilibrium and the net torque is zero about a given point, it must be zero at any other point.

In order to prove the above statement, we consider a simple example of a uniform bar of length 'l' resting on two pegs at its ends as shown in the Fig.5.11. The bar is in equilibrium under the action of three forces that is the weight of the bar $\vec{W}$ and the forces $\vec{F_1}$ and $\vec{F_2}$ exerted by the pegs in the upward direction as shown in the figure.

Applying the first condition of equilibrium we get

$\Sigma F_x = 0$ (No forces acting along horizontal)

$$\Sigma F_y = 0$$

$$F_1 + F_2 - W = 0$$

To write the torque equation we must choose an axis. Let this axis pass through the point 'O' the centre of the bar. Since the bar is at rest we apply the second condition of equilibrium, taking

the moment about the point O. Thus we have

$$\Sigma \tau = 0$$

$$F_1 \times \frac{l}{2} - F_2 \times \frac{l}{2} + W \times 0 = 0$$

$$F_1 - F_2 = 0$$

Now we take the moment about an axis passing through a point P lying to the right of 'O' at a distance x from it.

Now the sum of the moments of all the three forces is

$$\Sigma \tau = F_1 \left(\frac{l}{2} - x\right) - F_2 \left(\frac{l}{2} + x\right) + W \times x$$

$$= (F_1 - F_2)\frac{l}{2} - (F_1 - F_2 - W)\, x$$

since $F_1 - F_2 = 0$ and $F_1 + F_2 - W = 0$

$$\Sigma \tau = 0$$

which proves the statement. We are thus independent in choosing the axis about which the moment is to be taken. This is an important result because this provides a suitable choice for the location of the axis. By this we mean that one should try to select such an axis for which

(i) The unknown forces, if any, may be inactive while taking the moment. This is possible only if the moment arms of these forces are zero.

(ii) The forces involved may be minimum in number so as to ease the calculation work.

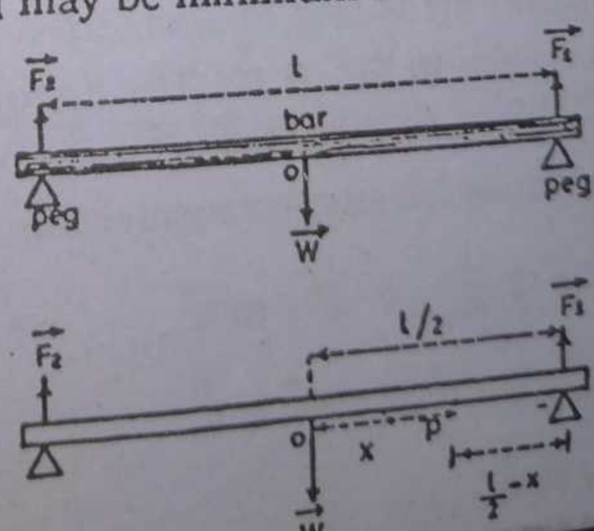


Fig. 5.11 Location of an axis.

5.7 ANGULAR MOMENTUM

A body having translatory motion possesses linear velocity and linear momentum. In the same way a body having rotatory motion possesses angular velocity and angular momentum. Thus the angular momentum is associated with angular (rotatory) motion. This means, in rotatory motion angular momentum plays a role which is completely analogous to that played by linear momentum in translatory motion. Angular momentum, like linear momentum obeys the law of conservation.

In order to have a tangible concept of angular momentum, we first discuss the angular momentum of a particle. Consider a particle of mass 'm'. Let $\vec{r}$ be its position vector and $\vec{p}$ be the linear momentum measured in an inertial frame of reference with origin 'o' as shown in Fig. 5.12. The angular momentum of the particle about the origin 'O' is defined as the vector product of $\vec{r}$ and $\vec{p}$. Thus if $\vec{l}$ stands for angular momentum, then

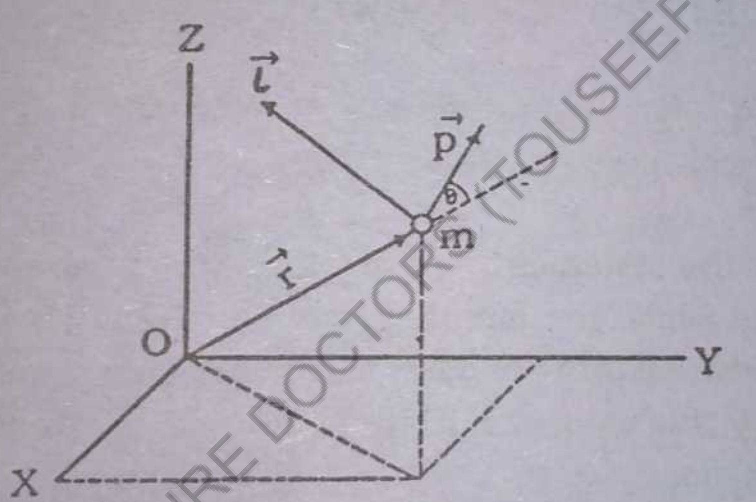


Fig. 5.12 Illustrating the definition of the angular momentum $\vec{l}$ of a particle about the origin o. With $\vec{l} = \vec{r} \times \vec{p}$ we see that the vector $\vec{l}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\vartheta} \tag{5.11}$$

$$= m\vec{r} \times \vec{\vartheta}$$

where

$$\vec{p} = m\vec{\vartheta}$$

and

$\vec{\vartheta}$ represents the velocity of the particle.

Since the vector product of two vectors is itself a vector, so the angular momentum is a vector. Its direction lies along the normal to the plane formed by the vectors $\vec{r}$ and $\vec{p}$ as given by the right-hand rule. The magnitude of the angular momentum is given by

$$|\vec{l}| = l = (r)(p)\sin\theta$$

where r and p represent magnitude of $\vec{r}$ and $\vec{p}$ repectively

$$l = m\vartheta r \sin\theta \qquad 5.12(a)$$

Where θ is the angle between $\vec{r}$ and $\vec{p}$. In circular motion $\vec{r}$ and $\vec{p}$ are perpendicular to each other, so the measure of θ is 90° and $\sin 90^\circ = 1$

Thus for circular motion we have

$$|\vec{l}| = l = (r)(p)(1) = rp \qquad 5.12(b)$$

If x,y,z are the components of $\vec{r}$ and p_x , p_y , p_z are the components of $\vec{p}$.then using the definition of vector product we write

$$\vec{l} = \vec{r} \times \vec{p}$$

$$= (xi + yj + zk) \times (p_x i + p_y j + p_z k)$$

$$= (yp_z - zp_y)\,i + (zp_x - xp_z)\,j + (xp_y - yp_x)\,k \qquad 5.13$$

The above expression for $\vec{l}$ can be written in the determinant form as

$$\vec{l} = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

The scalar components of angular momentum $\vec{l}$ are

$$l_x = yp_z - zp_y$$

$$l_y = zp_x - xp_z$$

$$l_z = xp_y - yp_x$$

The dimensions of angular momentum are given below

$$[l] = [r]\,[p]$$

$$= [r]\,[m]\,[\vartheta]$$

$$= L.\,M.\,\frac{L}{T}$$

$$= L^2\,M\,T^{-1}$$

In the system international (SI), the unit of length is the metre (m), the unit of mass is the kilogram (kg) and the unit of time is the second (s). Hence the units of angular momentum are

$$\frac{(m^2)\,(kg)}{s} = \frac{(kg)\,(m)}{s^2} \times (m)\;(s)$$

$$= (N)\;(m)\;(s)$$

$$= Js$$

Thus the units of angular momentum in the S I system are Joule-second (Js).

Example 5.5

A particle of mass 400 gram rotates in a circular orbit of radius 20 cm at a constant rate of 1.5 revolutions per second. Evaluate the angular momentum of the particle with respect to the centre of the orbit.

Solution:

As is clear from Fig. 5.13, the angular momentum $\vec{l}$ which is normal to the plane of the orbit is parallel to the angular velocity

$\vec{\omega}$ which lies along the axis of rotation. Using Eq. (5.12) the magnitude of the angular momentum $\vec{l}$ is

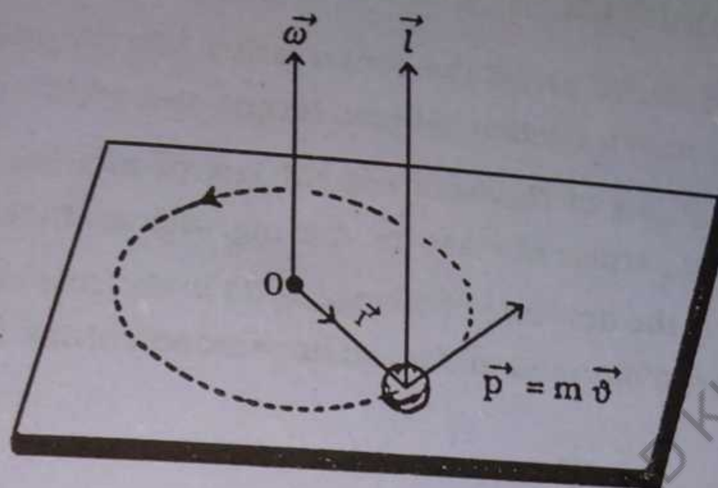


Fig: 5.13

$l = m \vartheta r \sin\theta$

$= m r^2 \omega \sin\theta$

Where

$\vartheta = r\omega$

$m = 400 \text{ gm} = 0.4 \text{ kg}$

$r = 20 \text{ cm} = 0.2 \text{ metre}$

$\theta = 90^\circ$

$l = 0.4 \times (0.2)^2 \times \omega \sin 90^\circ$

$= 0.4 \times 0.04 \times \omega \times 1$

$= 0.016\,\omega$

But $\omega = 2\pi f$

Where

f = no of revolutions/sec = 1.5 rev /sec

$l = 0.016 \times 2\pi \times 1.5$

$= 0.151$ J.S

5.9 CONSERVATION OF ANGULAR MOMENTUM OF A PARTICLE

In order to derive the conservation law for angular momentum, we obtain a relation between torque and angular momentum.

According to Newton's second law of motion, the net force acting on a particle of mass 'm' moving with an instantaneous velocity $\vec{v}$ is the time rate of change of its linear momentum. Thus if $\vec{F}$ stands for force and $\vec{p}$ for the linear momentum, then

$$\vec{F} = \frac{d}{dt}(\vec{p}) \qquad 5.14$$

Taking the vector product of both the sides of the above equation with r from the left we get

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \qquad 5.15$$

But by definition $\vec{r} \times \vec{F}$ is the torque acting on the particle.

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Also by definition, the angular momentum is

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating the above equation with respect to time, we get

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

where

$$\vec{v} = \frac{d\vec{r}}{dt}$$

and $\vec{p} = m\vec{v}$

$$\frac{d\vec{l}}{dt} = m\ \vec{v} \times \vec{v} + \vec{\tau}$$

$$= 0 + \vec{\tau}$$

Since the vector product of a vector with itself is zero

(i.e. $\vec{v} \times \vec{v} = 0$)

$$\therefore \quad \vec{\tau} = \frac{d\vec{l}}{dt} \qquad 5.16$$

This is the required relation. This equation states that the torque acting on a particle is the time rate of change of its angular momentum.

If the net external torque acting on the particle is zero, then Eq. 5.16, reduces to,

$$\frac{d\vec{l}}{dt} = 0 \qquad 5.17$$

$$\vec{l} = \text{constant} \qquad 5.18$$

Thus the angular momentum of a particle is conserved (constant) if the net torque acting on it is zero.

The conservation law obtained for single particle Eq.(5.18) can be extended to demonstrate conservation law for the angular momentum of a system of particles.

Consider a system of n particles which is acted upon by external as well as internal forces. We assume that the internal forces obey the law of action and reaction Hence they cancel out as a whole and the system is purely under the action of external (ap-

plied) forces. The total angular momentum of the system is given as:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + ... + \vec{l}_n \qquad 5.19\,(a)$$

$$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + ... + \vec{r}_n \times \vec{p}_n \qquad 5.19\,(b)$$

Taking the time derivative of both the sides of the above equation, we get

$$\therefore \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}_1 \times \vec{p}_1) + \frac{d}{dt}(\vec{r}_2 \times \vec{p}_2) + \frac{d}{dt}(\vec{r}_n \times \vec{p}_n)$$

$$= \vec{r}_1 \times \frac{d\vec{p}_1}{dt} + \vec{r}_2 \times \frac{d\vec{p}_2}{dt} + ... \vec{r}_n \times \frac{d\vec{p}_n}{dt}$$

$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + ------ + \vec{r}_n \times \vec{F}_n$$

$$= \vec{\tau}_1 + \vec{\tau}_2 + --- + \vec{\tau}_n \qquad 5.20(a)$$

$$= \sum_{i=1}^{n} \vec{\tau}_i = \vec{\tau} \qquad 5.20(b)$$

Where $\vec{F}_i$ and $\vec{\tau}_i$ are the external force and the external torque respectively acting on the i th particle.

If the net external torque $\vec{\tau}$ acting on the system is zero, then the total angular momentum ∵ (Eq. 5.19) is given by

$$\frac{d\vec{L}}{dt} = 0 \qquad 5.21$$

$$\therefore \quad \vec{L} = \text{constant} \qquad 5.22$$

Thus the total angular momentum of a system of particles is conserved (constant) if the net external (applied) torque acting on the system is zero.

Applications of the conservation law for angular momentum.

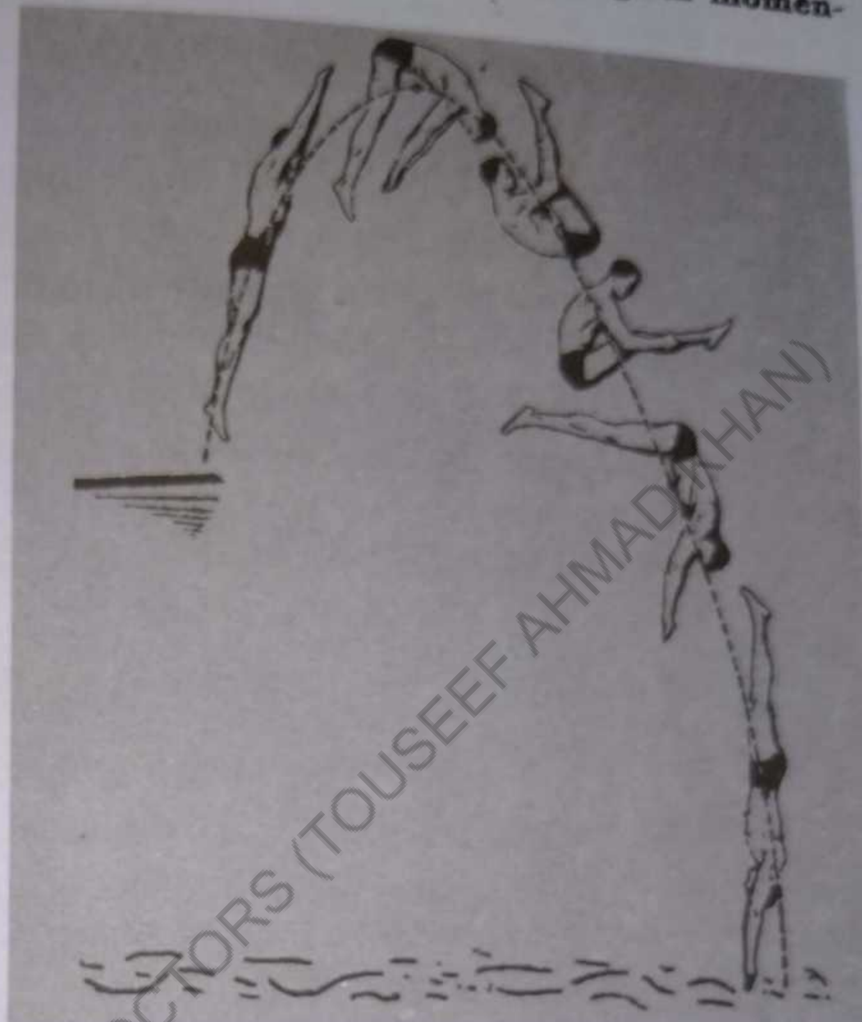


Fig. 5.14

(i) You might have seen a figure skater under going a spin motion in the finale of an act. We assume that there is no friction between the skater and the ice and hence there are no external torques acting on the skater. Thus the angular momentum I ω of the skater which is the product of moment of inertia, I, of the skater and the angular velocity, ω, of the skater, is constant, since the moment of inertia depends on the distribution of mass, the skater can decrease his or her moment of inertia by pulling his or her hands and feet close to the body. As a result the angular velocity of the skater increases and so the spinning takes place at a rapid rate.

(ii) When divers and acrobats who wish to make several somer saults, they pull their hands and feet close to their bodies in

order to rotate at a higher rate. Due to the close distribution of mass the moment of inertia decreases. This causes an increase in the angular velocity enabling them to make several somer saults.

In these examples the external force due to gravity acts through the centre of gravity and hence exerts no external torque about this point. Therefore, the angular momentum about the centre of gravity must be constant. Thus when divers and figure skaters wish to double their angular velocity of their spin motion, they must reduce their moment of inertia to the half its initial value by pulling their hands and feet close to their bodies.

Objective questions

1.

A) Fill in the blanks with appropriate words:

(i) The physical quantity which tends to rotate a body is called ....

(ii) In the system international, the units of torque are .....

(iii) Conventionally the torque, producing clock wise rotation is taken as ...

(iv) The angular momentum of a body is conserved if the net ... on it is zero.

(v) If the net torque on a body is zero, it is said to be in ... equilibrium.

(vi) Two forces which cannot be replaced by a single equivalent force are said to form a .....

(vii) ...... is defined as the time rate of change of angular momentum.

(viii) In the system international the units of angular momentum are ....

(ix) The angular momentum of an isolated system is ...

(x) A body is said to be in .... equilibrium if the net force on it is zero.

(B) Write "True" or "False" against each statement.

(i) Torque is a vector product of two appropriate vectors.

(ii) Angular momentum is a scalar product of two appropriate vectors.

(iii) The magnitude and direction of angular momentum depend on the choice of origin.

(iv) A particle moving along a line that passes through the origin has zero angular moment about that origin.

(v) The angular momentum of a body which is under the action of central forces is constant.

(vi) The direction of angular velocity is taken to be the same as that of angular momentum.

(vii) The torque due to action - reaction force is not zero.

(viii) Angular acceleration is in the same direction as the angular velocity if the latter is decreasing with time.

(ix) An electron of an atom possesses only orbital angular momentum.

(x) The orbital angular momentum of a body is its angular momentum about an axis passing through the body itself.

(xi) The centre of mass of a body coincides with its centre of gravity irrespective of its mass and size.

(C) Tick (✓) the correct answer.

(i) Torque is defined as

(a) time rate of change of angular momentum.

(b) time rate of change of linear momentum.

(c) time rate of change of angular velocity.

(ii) The vector quantity torque

(a) depends on the choice of origin.

(b) does not depend on the choice of origin.

(c) rotates a body always in clock wise.

(iii) Every point of a rotating rigid body has

(a) the same angular velocity.

(b) the same linear velocity.

(c) the same linear acceleration.

(iv) The right hand rule is applied to find

(a) the direction of a vector obtained by the vector product of two vectors.

(b) the magnitude of a vector obtained in the above manner.

(c) neither the direction nor the magnitude

(v) Two forces which form a couple

(a) can be replaced by a single equivalent force.

(b) cannot be replaced by a single equivalent force.

(c) are perpendicular to each other.

(vi) The direction of torque is

(a) the same as the direction of the corresponding applied force.

(b) opposite to the direction of the applied force.

(c) perpendicular to the direction of the applied force.

(vii) The centre of mass of a system of particles

(a) coincides always with the centre of gravity.

(b) never coincides with the centre of gravity.

(c) coincides with the centre of gravity only in a uniform gravitational field.

Questions

1. Is it possible to calculate the torque acting on rigid body without specifying the axis ?

(2) Is torque independent of location of axis ?

(3) In the expression for torque $\vec{\tau} = \vec{r} \times \vec{F}$, is $\vec{r}$ equal to the moment arm? Explain.

(4) If a particle moves in a straight line, is its angular momentum zero with respect to an arbitrary axis ? Is its

angular momentum zero with respect to any specified origin? Explain.

(5) If a torque on a particle about an arbitrary axis is zero, what can you say about its angular momentum, about that origin.

(6) If the linear velocity of a particle is constant in time, can its angular momentum be constant in time about an arbitrary axis ?

(7) A particle moves in a straight line and you are told that the torque acting on it is zero about some specified origin. Does this necessarily mean that the net force acting on the particle is zero.Can you conclude that its linear velocity is constant? Explain

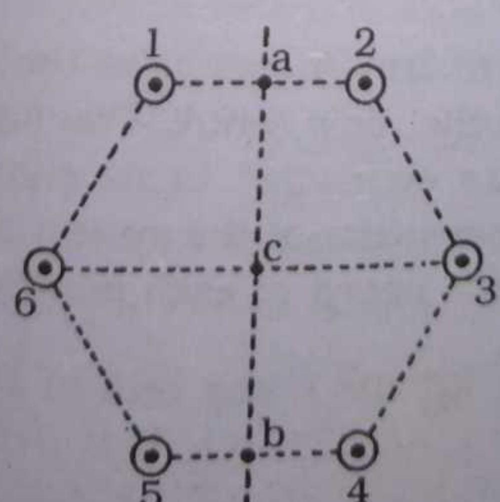
(8) Suppose that the velocity of a particle is completely specified. What can you say about the direction of its angular momentum vector with respect to the direction of motion?

(9) Why is it easier to keep your balance on a moving bicycle than on a bicycle at rest?

(10) A projectile is fired into air and suddenly explodes into several fragments. What can be said about the motion of the fragments after explosion?

Problems:

1. Locate the centre of mass of a system of particles each of mass 'm', arranged to correspond in position to the six corner of a regular (planar) hexagon.



(Ans) Centre of the hexagon

2. Find the position of centre of mass of five equal-mass particles located at the five corners of a square- based right pyramid with sides of length 'l' and altitude 'h'.

(Ans $\frac{1}{5}$ h from base along the line perpendicular to the base and passing through peak).

3. The mass of the sun is 329.390 times the earth's mass and the mean distance from the centre of the sun to the centre of the earth is 1.496×10^8 km. Treating the earth and sun as particles with each mass concentrated at the respective geometric centre, how far from the centre of the sun is the C.M (centre of mass) of the earth-sun system? Compare this distance with the mean radius of the sun (6.9960×10^5 km)

(Ans. 4.54×10^2 km; 6.48×10^{-4}).

4. A particle with mass 4 kg moves along the x-axis with a velocity v = 15 t m/sec, where t = 0 is the instant that the particle is at the origin.

(a) At t = 2sec, what is the angular momentum of the particle about a point P located on the + y-axis, 6m from the origin? (b) What torque about P acts on the pa ticle?

(Ans. (a) 720 kg m^2/sec; (b) 360 N.m).

5. A particle of mass 'm' is located at the vector position $\vec{r}$ and has a linear momentum vector $\vec{p}$. The vectors $\vec{r}$ and $\vec{p}$ are non zero. If the particle moves only in the x, y plane, prove that

$L_x = L_y = 0$ and $L_z \neq 0$

6. A light rigid rod 1 m in length rotates in the xy-plane about a pivot through the rod's centre. Two particles of mass 2kg and 3kg are connected to its ends. Determined the angular momentum of the system about the origin at the instant the speed of each particle is 5m/sec.

(Ans: 12.5 kg m² / see out of the plane)

7. A uniform beam of mass 'M' supports two masses m_1 and m_2. If the knife edge of the support is under the beam's centre of gravity and m_1 is at a distance 'd' from the centre, determine the position of m_2 such that the system is balanced.

(Ans. $^{m_1}/_{m_2}\, d$)

8. A uniform ladder of length l and weight W=50N rests against a smooth vertical wall. If the coefficient of friction between the ladder and the ground is 0.40, find the minimum angle θ_{min} such that the ladder may not slip.

(Ans. $\theta_{min} = 51.3°$)

9. A ladder with a uniform density and a mass 'm' rests against a frictionless vertical wall at an angle of 60°. The lower end rests on a flat surface where the coefficient of friction (static) is 0.40. A student with a mass (M = 2m) attempts to climb the ladder. What fraction of the length 'L' of the ladder will the student have reached when the ladder begins to slip?

(Ans. 0.789)

10. A particle of mass 0.3 kg moves in the xy-plane. At the instant its coordinates are (2,4) m, its velocity is $(3\hat{i} + 4\hat{j})$ m/sec. At this instant determine the angular momentum of the particle relative to the origin.

(Ans. $-1.2\hat{k}$ J.S)

11. A uniform horizontal beam of length 8 m and weighing 200N is pivoted at the wall with its far end supported by a cable that makes an angle of 53° with the horizontal. If a person weighing 600N stands 2m from the wall, find the tension and the reaction force at the pivot.

Ans. $\left\{\begin{array}{lll} T & = & \text{tension} = 313\text{N} \\ R & = & \text{reaction force} = 581\text{N} \\ \theta & = & \text{angle made by the reaction force with the horizontal} \\ & = & 71.1° \end{array}\right\}$

Chapter: 6

Gravitation

INTRODUCTION

All of Isaac Newton's efforts in mechanics were focussed toward the explanation of the motions of the planets around the sun and of the moon round the earth. In 1666, Newton prompted by a simple observation (fall of an apple toward the earth) concluded that the force that caused an apple to fall to the ground and the force that kept the moon in its orbit around the earth were only different manifestation of one universal force called force of Gravitation. From the above assumptions, Newton made the hypothesis that every body in the universe exerts a gravitational force on every other body. This gravitational force is responsible for the motion of the planets around the sun, the motion of falling bodies towards earth, etc.

6.1 NEWTON'S LAW OF UNIVERSAL GRAVITATION

In order to explain the gravitational force Newton formulated the law of universal gravitation as under:

Newton began on the basis of approximation that the moon's orbit is circular, he then calculated centripetal acceleration of the moon, a_m, about the earth applying Huygen's formula for centripetal acceleration in the form

$$\vec{a}_c = -\frac{V^2}{r}\hat{r} \qquad 6.1(a)$$

Where $\hat{r}$ is a unit vector directed from the centre of the circle to the instantaneous location of the moving body. The minus sign in the equation specifies that the direction of $\vec{a}_c$ is inward (towards

the centre of the circle).

The magnitude of the centripetal acceleration is then given by

$$|\vec{a}_c| = a_c = \frac{V^2}{r} = \frac{4\pi^2}{T^2}r \qquad 6.1(b)$$

$$\because \quad V = \frac{2\pi r}{T}$$

Consequently, we can rewrite the above equation for the magnitude of the centripetal acceleration of the moon.

$$a_m = \frac{4\pi^2}{T^2}R_m \qquad 6.1(c)$$

where

$R_m = 3.84 \times 10^5$ km and represents the distance from the centre of the earth to the centre of the moon and called orbit radius of the moon.

$T = 27.3$ days $= 2.36 \times 10^6$ s and represents the time taken by the moon to complete one revolution around the earth.

Substituting the values of R_m and T in Eq. 6.1(c), the magnitude of the centripetal acceleration of the moon is found to be

$$a_m = 2.72 \times 10^{-3}\ ms^{-2} \qquad 6.2$$

The magnitude of the centripetal acceleration of the moon given by Eq.6.2 provided the basis for the mathematical formulation of the law of universal gravitation. If the centripetal acceleration is a gravitational acceleration, the apple would also experience an acceleration of magnitude $a_m = 2.72 \times 10^{-3}\ ms^{-2}$, provided the apple is placed at a distance of $R_m = 3.84 \times 10^8$ m from the centre of the earth.

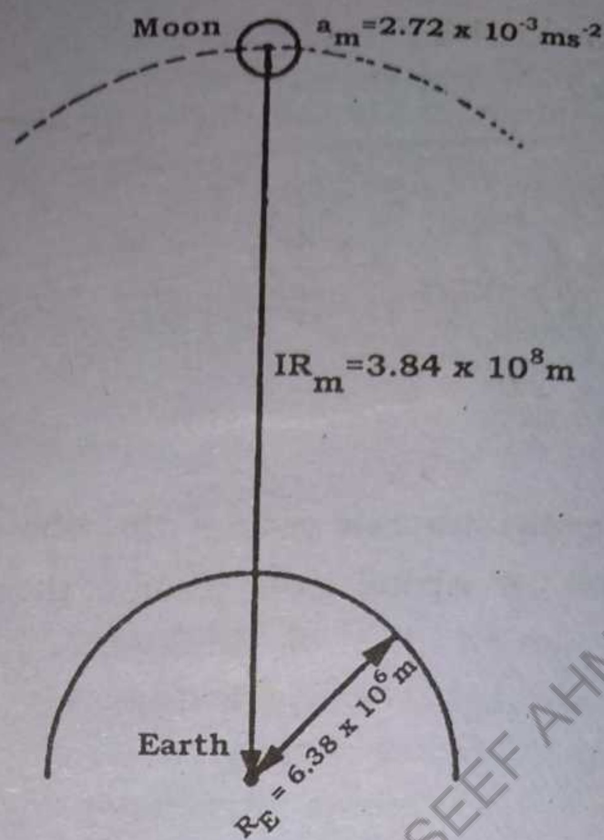


Fig: 6.1 Gravitational acceleration of apple decreases when it is moved away from the earth. Moon and apple have same acceleration when at the same distance.

We know that near the surface of the earth an object falling freely experiences an acceleration of 9.8 ms^{-2} the value of a_m is then approximately equal to $1/(60)^2$ that of the acceleration due to gravity $g = 9.8\ ms^{-2}$ at the surface of the earth.

that is,

$$\frac{a_m}{g} = \frac{1}{(60)^2} \qquad 6.3$$

From Fig. 6.1, we see

$$\frac{R_E^2}{R_m^2} = \frac{(6.38 \times 10^3\ km)^2}{(3.84 \times 10^5\ km)^2} = \frac{1}{(60)^2} \qquad 6.4$$

It appears that

$$\frac{a_m}{g} = \frac{R_E^2}{R_m^2} \qquad 6.5$$

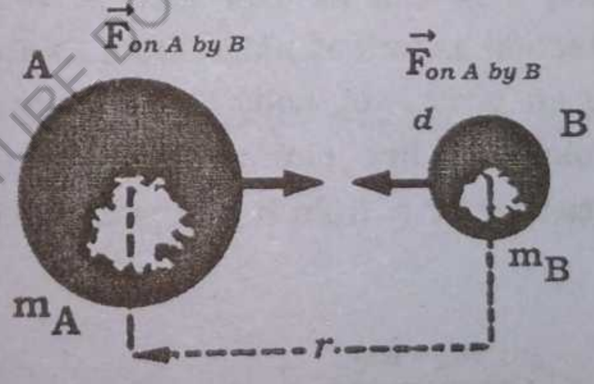
Where R_E is the distance from the centre of the earth to its surface i.e the radius of the earth as shown in Fig. (6.1). Comparison of the gravitational acceleration of the apple falling from the tree with the gravitational acceleration of the moon (if centripetal acceleration is the gravitational acceleration) falling in its orbit suggests that the acceleration of the moon because of the gravitational force exerted on the moon by the earth, depends inversely as the square of the distance from the centre of the earth to the centre of the moon. To make the law a truly universal law of gravitation it must apply to any pair of bodies A and B and not just to the earth and the apple, or the earth and the moon.

The magnitude $F_{\text{on A by B}}$ of the gravitational force $\vec{F}_{\text{on A by B}}$ exerted on body A due to the presence of the body B is given in the form

$$F_{\text{on A by B}} \propto \frac{1}{(r_{\text{From B to A}})^2} \qquad 6.6$$

Where $\vec{r}_{\text{From B to A}}$ describes the position of body A with respect to another body B as shown in Fig. 6.2

The magnitude, $F_{\text{on A by B}}$, of the gravitational force $\vec{F}_{\text{on A by B}}$ must also be proportional to the mass m_A of the body on which it is acting.



r *From B to A*

Fig: 6.2 gravitational force between m_A and m_B

therefore

$$F_{\text{on A by B}} \propto m_A \qquad 6.7$$

As the bodies A and B are chosen arbitrarily, there must be a force $\vec{F}_{\text{on B by A}}$ due to the presence of the body A, therefore.

$$F_{on\ B\ by\ A} \propto m_B \tag{6.8}$$

where m_B is the mass of body B.

According to the third law of motion, the magnitudes of two forces are equal, i.e

$$F_{on\ A\ by\ B} = F_{on\ B\ by\ A}$$

The proportionality Eq. 6.7 can therefore be written as

$$F_{on\ A\ by\ B} \propto m_A$$

when proportionalities are taken together

$$F_{on\ A\ by\ B} \propto m_A m_B \tag{6.9}$$

Combining Eq. 6.6 & Eq. 6.9, we get

$$F_{on\ A\ by\ B} \propto m_A\ m_B \,/\, (r_{from\ B\ to\ A})^2$$

When written in form of an equation

$$F_{on\ A\ by\ B} = G\frac{m_A m_B}{(r_{from\ B\ to\ A})^2} \tag{6.10}$$

Eq. 6.10 gives the magnitude of the gravitational force and G is a universal gravitational constant ($G = 6.673 \times 10^{-11}\ N\text{-}m^2/kg^2$)

The equation 6.10 can be expressed in vector form which then gives the direction as well as magnitude. As shown in Fig. 6.2. The force exerted on body A by body B is always directed toward body B and acts along the line joining their centres and vice versa. As the displacement vector $\vec{r}$ from B to A is directed from the body B to A, we have

$$\vec{F}_{on\ A\ by\ B} = -G\frac{m_A \cdot m_B}{(r_{from\ B\ to\ A})^2}\ \hat{r}_{from\ B\ to\ A} \tag{6.11(a)}$$

The negative sign shows that the gravitational force is attractive.

Similarly, the force exerted by body A on body B is given by

$$\vec{F}_{on\ B\ by\ A} = -G\frac{m_B\ m_A}{(r_{from\ A\ to\ B})^2}\ \hat{r}_{from\ A\ to\ B} \tag{6.11(b)}$$

Comparing Eq. 6. 11 (a) with Eq. 6.11 (b) we note that

$(r_{\text{from B to A}})^2 = (r_{\text{from A to B}})^2$

and

$\hat{r}_{\text{from B to A}} = -\hat{r}_{\text{from A to B}}$

Consequently we have

$\vec{F}_{\text{on A by B}} = -\vec{F}_{\text{on B by A}}$

as shown in Fig. (6.2)

These two gravitational forces constitute an action - reaction pair in accordance with Newton's third law.

We may substitute shorthand notation for the subscripts such that

(i) $\vec{F}_{\text{from A to B}} = \vec{F}_{AB}$

(ii) $\vec{F}_{\text{from A to B}} = \vec{F}_{BA}$

(iii) $\vec{r}_{\text{from A to B}} = \vec{r}_{BA}$

(iv) $\vec{r}_{\text{from A to B}} = \vec{r}_{AB}$

consequently, we rewrite Eq. 6.10, Eq. 6.11 (a) and Eq. 6.11 (b) in the form

$$F_{AB} = G\frac{m_A m_B}{r_{BA}^2} \qquad 6.12(a)$$

$$\vec{F}_{AB} = -G\frac{m_A m_B}{r_{BA}^2}\hat{r}_{BA} \qquad 6.12(b)$$

$$\vec{F}_{BA} = -G\frac{m_B m_A}{r_{AB}^2}\hat{r}_{AB} \qquad 6.12(c)$$

The Eq. (6.11) and Eq. (6.12) define Newton's universal law of gravitation. It is truly a universal law, since it describes the gravitational interaction between pairs of objects wherever we ob-

serve them through out the universe. The law states that

"Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them and directed along the line joining their centres".

The value of the gravitational constant, G, was determined experimentally. The first accurate measurement was made by Lord Cavendish in 1798. Later in ninteenth century significant improvement were made by Poynting and Boys. The present accepted value of the universal gravitational constant G was obtained by P.R.Heyl and P. Chizano Wski at the U.S National Bureau of standards, i.e

$$G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2$$

Example: 6.1

Compute the gravitational force of attraction between two balls each weighing 5kg, when placed at a distance of 0.33 m apart.

$$F = G\frac{m_1 m_2}{r^2}$$

$$= \frac{6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \times 5\text{kg} \times 5\text{kg}}{(0.33\text{m})^2}$$

$$= 1.5 \times 10^{-8} \text{ N}$$

6.2 MASS AND AVERAGE DENSITY OF EARTH

Mass of the earth can be determined by using the Newton's law of universal gravitation. Consider an object of mass M placed near the surface of the earth. If M_E is the mass of the earth and R_E its radius, then the magnitude of the gravitational force with which the earth attracts the object towards its centre, is given by

$$F = G\frac{MM_E}{R_E^2} \qquad 6.13$$

The force on the object is also given by the mass M of the object multiplied by the acceleration due to gravity which is equal to its weight W. (Second law of motion)

Hence $F = W = Mg$ 6.14

Where g is acceleration due to gravity. Thus from Eq. 6.13 and Eq. 6.14 we get

$$G\frac{MM_E}{R_E^2} = Mg$$

Solving for M_E we get

$$M_E = \frac{R_E^2 g}{G} \qquad 6.15$$

Since $R_E = 6.38 \times 10^6$ m, $g = 9.8ms^{-2}$ and $G = 6.67 \times 10^{-11}$ N-m^2 kg^{-2}

Therefore

$$M_E = \frac{9.8\,(ms^{-2}) \times (6.38 \times 10^6 m)^2}{6.67 \times 10^{-11}\,(N\text{-}m^2\,kg^{-2})}$$

$$= 5.98 \times 10^{24}\ kg$$

The average density or mass per unit volume, ρ, of the Earth is given by

$$\rho = \frac{M_E}{V} \qquad 6.16$$

Where V is the volume of the earth

$$V = \frac{4}{3}\pi R_E^3 \qquad 6.17$$

Substituting for M_E and V in Eq. 6.16 we can obtain the average density of the earth which is $5.5 \times 10^3\ kg/m^3$

EXAMPLE: 6.2

Calculate the mass of sun, when earth is orbiting around it. Let M be the mass of the sun and m be the mass of the earth

orbiting around the sun. Let R be the distance between the centres of the earth and the sun and is called orbit radius of earth.

The magnitude of the force of attraction between the sun and the earth is given by Eq 6.10.

$$F = G\frac{Mm}{R^2}$$

The centripetal acceleration of the earth

$$a_c = \frac{4\pi^2 R}{T^2}$$

Where T is period of revolution of earth around sun. (365.30 days)

From Newton's second law of motion

$$F = ma = m\frac{4\pi^2 R}{T^2}$$

$\therefore$

$$G\frac{Mm}{R^2} = m\left(\frac{4\pi^2 R}{T^2}\right)$$

Hence

$$M = \frac{4\pi^2 R^3}{GT^2}$$

From astronomical data, we know.

$R = 1.49 \times 10^{11}\,m$

$G = 6.673 \times 10^{-11}\ N\text{-}m^2/kg^2.$

$T = 365.3 \times 24 \times 60 \times 60s$

Substituting the values of R,T and G in the above relation, we get the mass of the sun.

$M = 1.99 \times 10^{30}$ kg.

6.3 VARIATION OF 'g' WITH ALTITUDE AND DEPTH

The force of gravity on a body varies slightly from place to place on the earth for two reasons. First, the shape of the earth, and secondly, its rotation. The earth is not a perfect sphere, but bulges at the equator. Therefore if a body is taken from a pole to the equator its distance from the centre of the earth will change Consequently, in accordance with Newton's law of gravitation, the gravitational pull on it will also vary.

(a) Now let us consider the variation of g when a body moves distance upward or downward from the earth's surface. Let g be the value of acceleration due to gravity at the surface of the earth and g' at a height h above the surface of the earth. If the earth be considered as a sphere of homogeneous composition, then 'g' at any point above its surface will vary inversely as the square of the distance from that point to its centre. From Eq 6.15 we found that g is given by

$$g = \frac{M_E G}{R_E^2} \qquad 6.18$$

Similarly, the value of g' at a distance $(R_E + h)$ as shown in Fig. 6.3 is given by

$$g' = \frac{M_E G}{(R_E + h)^2}$$

therefore

$$\frac{g}{g'} = \frac{(R_E + h)^2}{R_E^2}$$

$$= 1 + \frac{2h}{R_E} + \frac{h^2}{R_E^2} + \ldots \qquad 6.19$$

If 'h' is small as compared to R_E, the radius of earth, the quantity h^2/R_E^2 in the above equation will be negligibly small and we shall therefore have

$$g' = g\left(1 - \frac{2h}{R_E}\right) \qquad 6.20$$

From this equation we can conclude that the greater the value of h, the smaller is the value of g' or the value of g decreases with altitude.

Table 6.1

variation of 'g' with altitude at 45° latitude.

Altitude (metres)	g, (metre / sec^2)	Altitude (metres)	g, (metre / sec^2)
0	9.806	32×10^3	9.71
1000	9.803	100×10^3	9.60
4000	9.794	500×10^3	8.53
8000	9.782	1000×10^3	7.41
16000	9.757	3800×10^5	0.00271

Fig. 6.3 Variation of 'g' with altitude.

(b) Let g' be the acceleration due to gravity at a depth d below the surface of earth as shown in Fig 6.4. That is at a distance $(R_E - d)$ from the centre of earth. From Eq 6.16, the mass of earth is given by.

$$M_E = \frac{4\pi}{3} R_E^3 \rho. \qquad 6.21$$

where ρ is the density of the earth supposed to be uniform everywhere.

The mass of the earth effective for attraction at a depth d from its surface is

$$M'_E = \frac{4\pi}{3} (R_E - d)^3 \rho \qquad 16.22$$

From Eq.6.18, the value of g at the surface is given by

$$g' = \frac{M_E G}{R_E^2} \qquad 6.23\ (a)$$

Putting the value of M_E in the above equation we get

$$g' = \frac{\frac{4\pi}{3} R_E^3 \rho G}{R_E^2}$$

$$= \frac{4\pi}{3} R_E \rho G \qquad 6.23\ (b)$$

At depth d the value of acceleration due to gravity is equal to

$$g' = \frac{M'_E G}{(R_E - d)^2} \qquad 6.24\ (a)$$

Putting the value of M'_E in the above equation we get

$$g' = \frac{\frac{4\pi}{3} (R_E - d)^3 \rho G}{(R_E - d)^2}$$

$$= \frac{4\pi}{3} (R_E - d) \rho G \qquad 6.24\ (b)$$

The ratio g' to g is given by dividing Eq. 6.24 (b) by Eq. 6.23 (b)

$$\frac{g'}{g} = \frac{R_E - d}{R_E}$$

$$= 1 - \frac{d}{R_E}$$

$$\therefore \quad g' = g\left(1 - \frac{d}{R_E}\right)$$

That is the value of g decreases with depth from surface of the earth. It also follows that at the centre of earth, when $d = R_E$, the value of g will be zero.

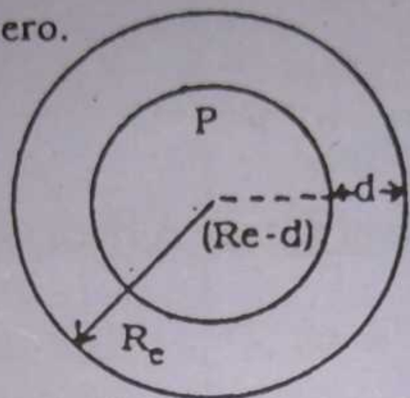


Fig. 6.4 Variation of 'g' due to depth.

It can be observed from both the cases that acceleration due to gravity decreases at a faster rate for a points above the surface of earth than for the same points below the surface of the earth.

The variation of 'g' in the earth's interior is shown in table 6.2

Table 6.2

Depth, km	g, (metre / sec^2)	Depth, km	g, (metre / sec^2)
0	9.82	1400	9.88
33	9.85	1600	9.86
100	9.89	1800	9.85
200	9.92	2000	9.86
300	9.95	2200	9.90
413	9.98	2400	9.98
600	10.01	2600	10.09
800	9.99	2800	10.26
1000	9.95	2900	10.37
1200	9.91	4000	8.00

6.4 WEIGHT

The weight of a body is the gravitational force exerted on it by the earth. Weight, being a force, is a vector quantity. The direction of this vector is the direction of the gravitational force that is toward the centre of the earth. The magnitude of the weight is expressed in the units of force, such as newton.

If an observer, stationed in a certain frame of references allows a body to fall, the body will fall with a certain acceleration. However, if we apply a suitable force, we can prevent it from accelerating. This force is a measure of the weight in the frame of reference of the observer. Thus we can define that weight of a body in a certain frame of reference is equal and opposite to the force required to prevent it from accelerating from rest in that frame of reference.

A force is measured by the acceleration, it imparts to a mass m. When a body of mass m is allowed to fall freely, its acceleration due to gravity is g and the force acting on it, is its weight W. According to Newton's second law of motion force is given by

$$F = ma$$

Replacing 'F' by 'W' and 'a' by 'g', in the above equation, we get

$$W = mg$$

The weight of a body is not a fixed quantity. It depends upon the location, as well as upon the motion of the frame of reference. The value of g i.e. the acceleration due to gravity, varies from place to place. Thus the weight of the body also varies from place to place, but its mass will remain the same, provided the velocity of the body is negligibly small as compared to the velocity of light.

6.5 WEIGHTLESSNESS IN SATELLITES

In connection with space exploration, one often encounters

about the weightlessness. It is well known fact that astronauts experience weightlessness when their space craft is orbiting around the earth. In such circumstances, the earth's gravitational pull is just balanced by the centripetal force resulting into weightlessness.

In order to understand the weightlessness in satellites, let us consider a simple case of the weight of a body in an elevator. Sup-

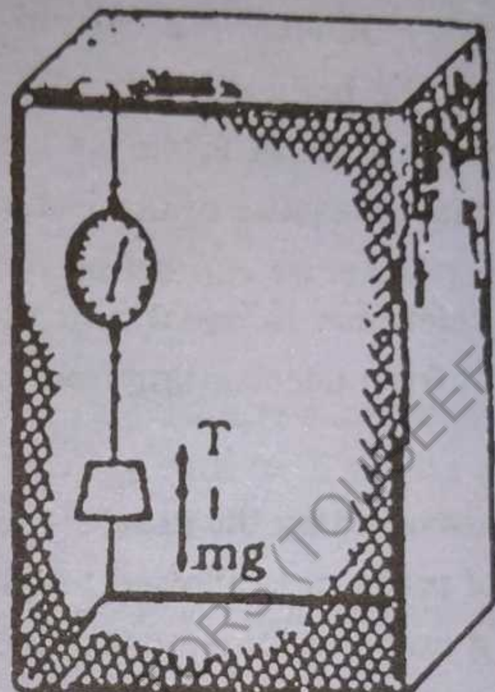


Fig:6.5 A body suspended in an elevator.

pose that a block is suspended from a spring balance by means of a thread attached to the ceiling of the elevator. Fig 6.5

(a) When the elevator is at rest, the force along the thread is equal and opposite to the force of gravity experienced by the block and this represents true weight of the block, i.e. $\vec{F} = m\vec{g}$

(b) When the elevator is ascending with an acceleration $\vec{a}$ relative to the earth, then the equation of motion will be

$$\vec{F}_w - m\vec{g} = m\vec{a} \qquad 6.25(a)$$

$$\vec{F}_w = m\vec{g} + m\vec{a} \qquad 6.25(b)$$

Eq 6.25 (b) gives apparent weight of block which is greater than true weight and the body feels "heavier".

(c) When the elevator is descending with the same accelera-

tion $\vec{a}$. the equation of motion will be

$$m\vec{g} - \vec{F}_w = m\vec{a}$$

$$\vec{F}_w = m\vec{g} - m\vec{a} \quad 6.26\ (a)$$

6.26 (b)

The Eq. 6.26 (b) gives apparent weight of block which is less than the true weight and the body appears "lighter"

(d) Suppose, if the cable supporting the elevator breaks, the elevator will fall down with an acceleration equal to g. The net force will be

$$\vec{F}_w = m\vec{g} - m\vec{g} \quad [\because \vec{a} = \vec{g}]$$

or

$$\vec{F}_w = 0 \quad 6.27$$

Consequently, the spring balance will read zero and the man in the elevator will find that the block has no weight besides the fact that the force of gravity still acts upon the block and its weight $\vec{w}$ is given by $m\vec{g}$. This is referred as the state of "weightlessness".

Our discussion of apparent weight can also be applied to the phenomenon of "weightlessness" in satellites. Bodies in an orbiting satellite are not weightless; the earth's gravitational attraction continues to act on them just as though they were at rest relative to the earth. But a space vehicle in orbit has an acceleration $a_\perp$ towards the earth's centre equal to the value of the acceleration due to gravity $\vec{g}$ at its orbit radius. The apparent weight $\vec{F}_w$ is given by.

$$\vec{F}_w = m\vec{g} - m\vec{a}_\perp$$

but in this case $\vec{g} = \vec{a}_\perp$

So $\vec{F}_w = 0$

It is in this sense that an astronaut or other body in the space craft is said to be "weightless" or in a state of "Zero g". The

earth's gravity acts on both the spacecraft and the bodies in it, giving each the same acceleration. Thus a body released in side the space craft does not fall relative to it, and it appears to be weightless.

6.6 ARTIFICIAL GRAVITY

In the preceding section, we have seen that all orbiting space crafts along with astronaut and other objects are in the state of free fall, and consequently will be in the state of weightlessness. Weightlessness in space crafts or satellites is highly inconvenient to an astronaut in many ways. For example, he cannot pour liquid into a cup, neither can he drink from it. If the space craft is a space laboratory intended to stay in space for a long period of time, the weightlessness may be a severe handicap to the astronaut in performing experiments. In order to overcome this problem an artificial gravity can be created in the space craft, by spinning it around its own axis. Fig 6.6

So that normal force of gravity can be supplied to the occupants in the spacecraft.

Consider a spacecraft consisting of two chambers connected by a tunnel of length 20 metres. Let us see how many revolutions per second must the space craft make in order to supply artificial gravity for the astronauts. Let T be the time for one revolution and υ be the frequency of rotation, than $\upsilon_t = 1$

The magnitude of centripetal acceleration in this case is given by

$$a_c = \frac{4\pi^2 R}{T^2} = 4\pi^2\upsilon^2 R$$

where R represents the half of the tunnel length i.e

$$R = \frac{20m}{2} = 10m.$$

$$\therefore \upsilon = \frac{1}{2\pi}\sqrt{\frac{a_c}{R}}$$

When a body falls freely $a_c = g$;

$$\therefore \upsilon = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$$

$$\therefore \upsilon = \frac{1}{2\pi}\sqrt{\frac{9.8ms^{-2}}{10m}}$$

$$= 0.158 \text{ rev/s}$$

υ = 9.5 revolutions/minute

Thus the astronaut should feel comfortable in space craft spinning at 9.5 revolution/minute at a distance of 10m away from the axis of rotation as shown in Fig. (6.6).

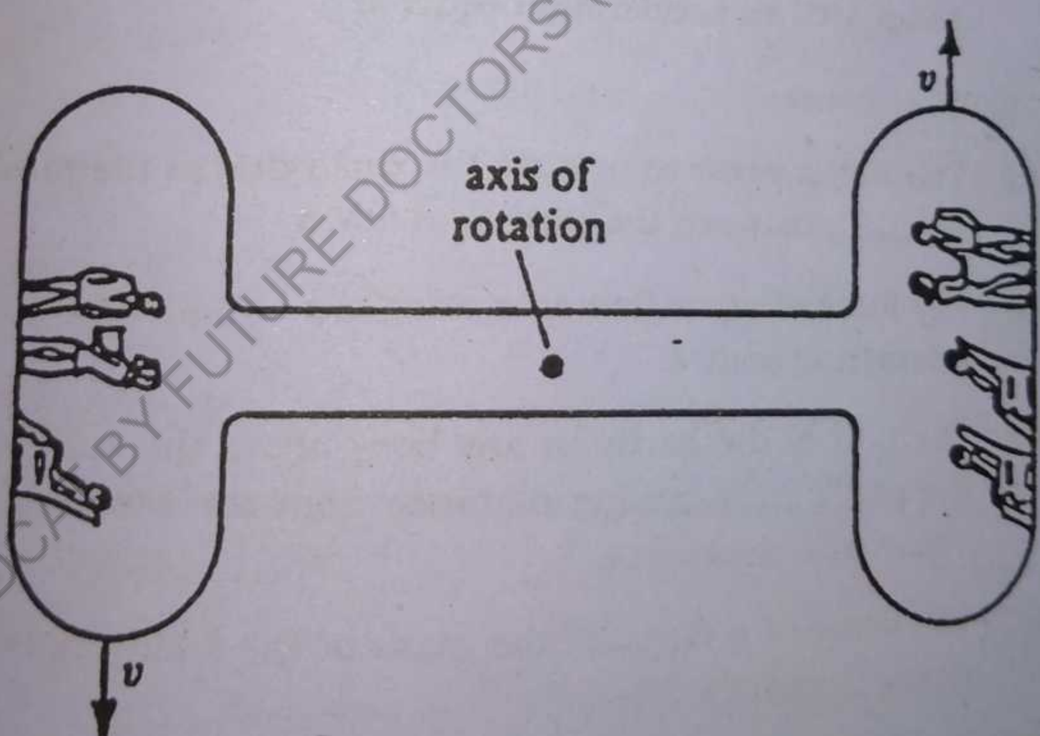


Fig: 6.6 *Rotating spaceship supplies "artificial" gravity for the passengers. Their apparent weight is the same as on earth.*

OBJECTIVE TYPE QUESTIONS

1 Whether true or false, if false correct the following:

a. Every body in the universe repels every other body with a force which is inversely proportional to the product of their masses.

b. The force of gravitational attraction is directly proportional to the square of the distance between two bodies.

c. The force with which the earth repels a body on its surface towards its centre is equal to its weight.

d. The value of g increases with the increase of the distance of the body from the centre of earth.

e. Gravitational force is of a very large order of about $6.67 \times 10^{-11}\ Nm^2/kg^2$.

f. If the cable carrying the elevator breaks, the elevator will go up with an acceleration equal to g.

2 Fill in the blanks

(a) The moon revolves around the earth due to the presence of .... between the earth and moon.

(b) The force of attraction acts along the .... joining the two interacting bodies.

(c) The pull of the earth on any body above the surface of the earth .... as the distance from the centre of the the earth decreases.

(d) The value of g is .... of the mass of the falling body towards the earth.

(e) Artificial gravity can be supplied by .... the space craft so that normal force of gravity can be supplied to the passengers.

3 Select the appropriate answer.

(a) The sun exerts a force of attraction on the planets thus keeping them in their:

(i) axes (ii) radii (iii) orbits (iv) State of motion

(b) The force with which the earth attracts the other body towards its centre is given by:

(i) $F = \dfrac{M_E m}{R_E^2}$ (ii) $F = G\dfrac{M_E m}{R_E^2}$

(iii) $F = \dfrac{R_E^2 M_E m}{G}$

(c) Numerical value of G can also be estimated by knowing the :

(i) average density of earth (ii) circular motion

(iii) mass of earth.

(d) If we go away from the surface of the earth a distance equal to the radius of earth the value of g will become:

(i) one fourth (ii) one eighth (iii) one ninth

(e) Weight of a body is :

(i) vector (ii) scalar (iii) rotational quantity

QUESTIONS

1. State and explain the Newton's Universal Law of Gravitation.
2. Once value of G has been measured in the laboratory, how it can be used to determine the mass of the earth. Explain.
3. How one can determine the mass of Moon?
4. Explain how the value of g varies with a change in the altitude. What is the effect of depth on the value of g ?

5. Discuss the concept of weightlessness in the motion of space crafts.

6. Explain how artificial gravity is useful in space technology.

PROBLEMS

1. A 10 kg mass is at a distance of 1 m from a 100 kg mass. Find the gravitational force of attraction when
   (a) 10 kg mass exerts force on the 100 kg mass
   (b) 100 kg mass exerts force on the 10 kg mass

   (Ans. 6.67×10^{-8} N in both cases)

2. Compute gravitational acceleration at the surface of the planet Jupiter which has a diameter as 11 times as compared with that of the earth and a mass equal to 318 times that of earth.

   (Ans. 2.63 times that of earth)

3. The mass of the planet Jupiter is 1.9×10^{27} kg and that of the sun is 2.0×10^{30} kg. If the average distance between them is 7.8×10^{11}m, find the gravitational force of the sun on Jupiter.

   (Ans. 1.613 m/s^2: 69.8)

4. The radius of the moon is 27% of the earth's radius and its mass is 1.2% of the earth's mass. Find the acceleration due to gravity on the surface of the moon. How much will a 424N body weight there?

   (Ans. 1.613m/s^2 : 69.8 N)

5. What is the value of the gravitational acceleration at a distance of
   (i) earth's radius above the earth's surface?
   (ii) Twice earth's radius above the earth's surface.

   (Ans. 2.45 m/s^2, 1.09 m/s^2)

6. At what distance from the centre of the earth does the gravitational acceleration have one half the value that it has on the earth's surface?

(Ans. 1.41 earth's radius)

7. Compute the gravitational attraction between two college students of mass 80 and 50 kg respectively, 2m apart from each other. Is this force worth worrying about?

(Ans 6.67×10^{-8} N. Not at all)

8. Determine the gravitational attraction between the proton and the electron in a hydrogen atom, assuming that the electron describes a circular orbit with a radius of 0.53×10^{-10} m (mass of proton = 1.67×10^{-27} kg
mass of electron = 9.1×10^{-31} kg)

(Ans. 3.62×10^{-47} N)

9. A woman with a mass of 45 kg is standing on a scale in an elevator. The elevator accelerates upward with a constant acceleration of 1.2 m/s^2. What is the woman's weight as measured by her in the elevator.

(Ans: 495N)

Work Power and Energy

WORK AND ENERGY

In chapter 3 you have studied Newton's laws of motion,We can use these laws to write an equation for the acceleration of any body acted upon by any force. In addition in this we have techniques for finding the solution to that equation. The solution determine how the body moves when it has started in a particular manner. Thus the motion of almost any mechanical system can be studied by direct application of Newton's laws.

In practice, however, it is often much easier to study the mechanical problems by using other relations. A very important set of such relations involves a quantity called energy. However, it is important to realize that these energy relations are not independent of Newton's laws and,therefore, do not involve any new physical principle. In fact these energy relations re-express Newton's laws in a form which provides us much simpler analysis of a complex problem (problem involving a variable forces such as gravitational forces between bodies) than the direct application of Newton's laws.

Energy is the most important physical concept which is studied in all sciences. It is not a simple concept. We cannot define what energy is, yet we know what it can do, clear understanding of energy and its application was realized in 1847 when a German physicist Hermon Von Helmholtz enunciated the general law regarding the energy. Since then the consideration of energy in physical and biological processes has been a crucial ingredient in the effort to understand natural phenomena.

Closely associated with energy is the concept of work which provides a link between force and energy. Thus we first define work and making this as a foundation. We shall then be able to discuss Kinetic energy, potential energy and the law of conservation of energy and its application in various problems. A range of energies found in universe is shown in table 7.1.

Finally, we shall define power to characterize the rate of doing work or transfering energy in or out of a system.

Table 7.1 Range of energies found in universe

Joules		
10^{52}	—	Quasar explosion
10^{48}	—	
10^{44}	—	
10^{40}	—	Supernova explosion
10^{36}	—	
10^{32}	—	Sun's output in 1 year
10^{28}	—	Rotational energy of earth
10^{24}	—	Earth's annual energy from sun
10^{20}	—	Severe earthquake
10^{16}	—	H.bombs
10^{12}	—	First atomic bomb
10^{8}	—	Rocket launch
10^{4}	—	Lethal dose of x-radiation
1	—	Rifle bullet
10^{-4}	—	
10^{-8}		
10^{-12}	—	Fission of a uranium nucleus
10^{-16}	—	Electron in hydrogen atom chemical bond.
10^{-20}	—	

7.1 WORK

In our daily life, work is an activity that requires muscular or mental exertion. But in physics, the term has a distinctly different meaning, involving a force acting on a body while the body undergoes a displacement.

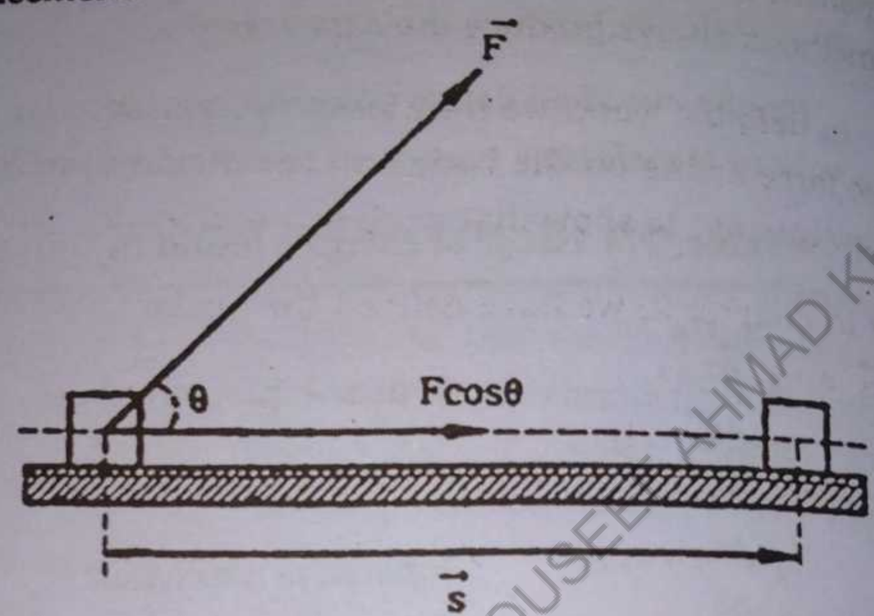


Fig. 7.1

The work done by a constant force, $\vec{F}$, is defined as the product of the component of the force in the direction of displacement, $\vec{S}$, and the magnitude of the displacement. Using Fig 7.1, we can express work done by the force analytically as

$$W = FS\cos\theta = (F\cos\theta)\,S \qquad 7.1\,(a)$$

Where

F - represents the magnitude of the vector $\vec{F}$

S - represents the magnitude of the vector $\vec{S}$

θ - is angle between $\vec{F}$ and $\vec{s}$.

Alternatively, the Eq. 7.1 can be written as

$$W = F(S\cos\theta) \qquad 7.1\,(b)$$

where $S\cos\theta$ is the component of displacement in the direction of $\vec{F}$. Thus work is also the product of the magnitude of force and the component of the displacement in the direction of $\vec{F}$.

The Eq. 7.1(a) and Eq.7.1(b) suggest that work can be calculated in two different ways: Either we multiply the magnitude of the force and the component of displacement in the direction of force or we multiply the magnitude of the displacement by the component of the force in the direction of the displacement. These two methods always produce the same result.

In defining work, we have used two vector quantities, namely, the force acting on the body and the displacement of the body. We now attempt to show that work is a scalar quantity.

In chapter 2, we have defined the scalar product of two vectors $\vec{a}$ and $\vec{b}$ as

$$c = \vec{a} \cdot \vec{b}$$

where c is a scalar quantity which is given by

$$c = a\, b \cos\theta$$

Here θ is angle between two vectors $\vec{a}$ and $\vec{b}$. Applying the above definition of scalar product in this case, we immediately write

$$W = \vec{F} \cdot \vec{d}$$

Where W (work done by the force) is a scalar quantity by definition and is given by the product of the magnitude of the one vector by the component of the second vector in the direction of the first vector. That is,

$$W = Fd \cos\theta$$

which is exactly the same as before.

Work is an algebraic quantity. It can be positive or negative depending on the value of angle between force, $\vec{F}$, and displacement, $\vec{S}$, We have the following important cases:

(i) When the component of the force is in the same direction of the displacement ($\theta = 0^\circ$), the work is positive. For example, when a spring is stretched, the work done by the stretching force is positive; when a body is lifted, the work done by the lifting force is positive.

(ii) When the direction of the force is opposite to the direction of the displacement ($\theta = 180^\circ$), the work is negative. For example, the work done by the gravitational force on the body being lifted is negative, since the (upward) displacement is opposite to the (downward) gravitational force.

(iii) When the force acts at right angles to the displacement ($\theta = 90^\circ$), the work is zero, i.e. the force does not produce work. For example, it is considered "hard work" to hold a heavy stone stationary at stretched hand, but no work is done in the technical sense. If a person walks along a level surface while carrying a box, no work is done, because the force has no component in the direction of the motion. When a body slides along a surface, the work done by the normal force acting on the body is zero. Also when a body moves in a circular path, the work done by the centripetal force on the body is zero because the centripetal force is always at right angles to the direction in which the body is moving.

Units of work

Work is a scalar quantity, and its unit is the unit of force multiplied by the unit of distance. In SI unit, the unit of work is 1 newton- metre (1 N-m). Another name for N-m is the Joule (J).

1 Joule = (1 newton) (metre)

1 J = 1 N-m

1055 Joule = 1 British Thermal Unit

1055 J = 1 BTU

In the physics of molecules, atoms and elementary particles, a much smaller unit is used. This unit is named, the electron volts (eV). $1\ eV = 1.60 \times 10^{-19}\ J$

Commonly, used multiples of the electron volt are

1 Million electron Volt = $1 MeV = 10^6\ eV$

1 Billion electron Volt = $1 BeV = 10^{12}\ eV$

7.2 WORK DONE AGAINST GRAVITATIONAL FORCE

We know that the force with which earth attracts a body towards its centre is called gravitational force which is equal to the weight of the body i.e mg; where m is the mass of the body and g is the acceleration due to gravity.

The gravitational force can do positive or negative work. When the body moves in the direction of the gravitational force i.e towards the earth, the work is done by the force of gravity on the body and it is positive, where as when the body moves against the direction of the gravitational force, the corresponding work done is negative.

Consider a ball of mass m, which is initially at a height h_i from the surface of the earth. The ball is moving upward and finally it reaches a height h_f, measured from the surface of the earth as shown in Fig. 7.2. During its upward motion, the only force acting on the body is the gravitational which is its weight mg. Here we

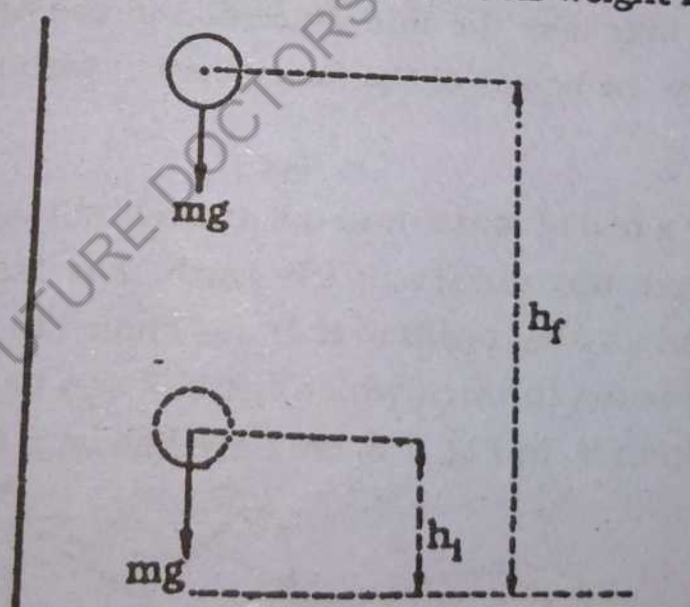


Fig. 7.2

assume that there is no force of friction. The displacement of the body is clearly $(h_f - h_i)=h$. Hence the work done is given by

$$W = \vec{F} \cdot \vec{s} = Fs\cos\theta = F(h_f - h_i)\cos\theta$$

$$= Fh\cos\theta$$

$$= Fh\cos 180°$$

$$= mgh\,(-1) \qquad [\because F = mg.\cos 180^\circ = -1]$$

$$W = -mgh \qquad 7.2$$

This is the work done against the gravitational force. The negative sign indicates that the force and displacements are oppositely directed. In this particular example one should note that the work done depends on the initial and the final positions (heights). But this result is quite general and is valid for any path (not necessarily straight up) joining the same initial and final positions.

This work done is stored in the body in the form of its Potential Energy. Thus

$$P.E = mgh$$

In the above expression for work against the gravitational force, only the difference of heights, between the two positions, appears. Hence it is not necessary to measure the vertical heights from the surface of the earth. The reference level may be chosen arbitrarily. We may take the initial position of the ball as the zero level to measure the height of the final position with respect to this zero level.

Consider a ball of mass m in our hand and take it to a height h measured from the surface of the earth. The initial position of the ball is A and its final position is B. (as shown in Fig. 7.3). There may be several paths through which the ball can be taken from position A to position B. In Fig. 7.3, we have shown a few of them.

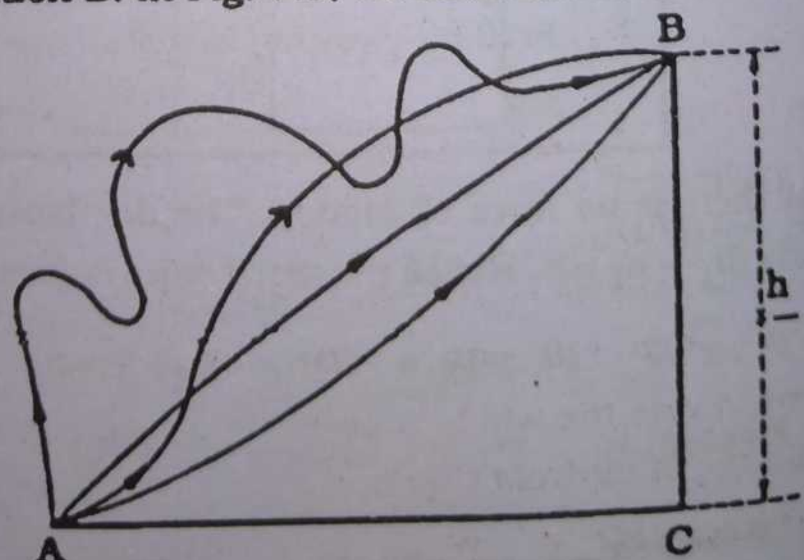


Fig. 7.3

Work done in moving the ball from position A to position B along any of the paths is the same and is equal to mgh. Thus the work-done in moving a body in a gravitational field is independent of the paths and depends only upon the initial and final positions (heights) of the ball.

To prove this statement, we consider a closed path of any shape in the gravitational field and show that the work done in carrying a body along this path is zero. For the sake of simplicity we take a triangular path ABCA in which the base BC is perpendicular to the gravitational field as shown in Fig. (7.4). The amount of work

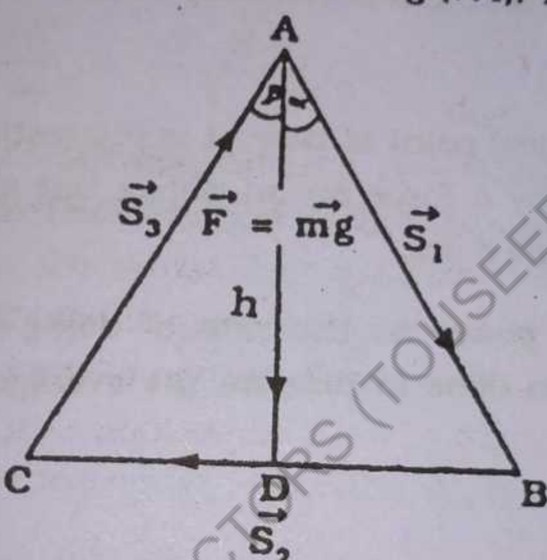


Fig. 7.4

done in carrying the body from A to B. B to C and from C to A are represented by $W_{A\to B}$, $W_{B\to C}$ and $W_{C\to A}$ respectively. Thus

$$W_{A\to B} = \vec{F}.\vec{S}_1 = (F)(S_1 \cos\alpha) = (mg)(h) = mgh$$

$$W_{B\to C} = \vec{F}.\vec{S}_2 = (F)(S_2 \cos 90) = (mg)(S_2 \times o) = 0$$

$$W_{C\to A} = \vec{F}.\vec{S}_3 = (F)(S_3 \cos(180-\beta)) = (mg)(-S_3\cos\beta) = -mgh.$$

where $h = m\overline{AD}$

Total work done along the closed path ABCA

$$= mgh + 0 - mgh = 0$$

We now divide the whole path into two parts, one from A to B, B to C and the other from C to A

$$\therefore \quad W_{A\to B\to C\to A} = W_{A\to B\to C} + W_{C\to A} = 0$$

Also $W_{C \to A} + W_{A \to C} = 0$

Comparing these equations

$$W_{A \to B \to C} = W_{A \to C}$$

Thus whether we carry the body from A to C (along AC directly) or along the path ABC, the work done is the same. There may be an infinite number of paths going from A to C, but the work done along any path is the same. Such a type of field of force in which the work is independent of the path is called a conservative field. Thus gravitational field is a conservative field.

7.3 POWER

From the practical point of view, it is interesting to know not only the work done by a force on an object but also the rate at which the work is being done.

We define power as the rate of doing work. When an amount of work ΔW is done in time Δt, the average power, P_{av}, is defined as

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad 7.3$$

In the above expression if $\Delta t \to o$, then limiting value of $\frac{\Delta W}{\Delta t}$ is called the instantaneous power at time t. The Eq. 7.3 is written as

$$P = \lim_{\Delta t \to o} \frac{\Delta W}{\Delta t} \qquad 7.4$$

In the International system of units, S I, the unit of work is 1 Joule and that of time is 1 second. So the corresponding unit of power is 1 Joule/sec, which is called 1 watt (W)

$1\ W = 1\ J/s = 1\ kg\ m^2/s^3$

1 megawatt (MW) = 10^6 W

1 gigawatt (GW) = 10^9 W

In the British Engineering system, the unit of power is 1 ft. lb/sec (Foot.Pound/second).

Since this unit is quite small, a bigger unit, called horse power, has been adopted and one horse power is equal to 746 watts.

We can express work as the product of power and time. The term kilo- watt-hour is originated from this definition of work. One kilo watt-hour is defined as the work done in one hour by an agency working at the constant rate of 1 kW that is 1000 joules per second. Since 1 hour=3600 seconds, so

1 kilowatt hour (1 kWh) $=1000 \times 3600$ joules$= 3.6 \times 10^6$ joules

We can obtain an alternative expression for power, using equation (7.3)

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{\vec{F}\cdot\vec{S}}{t}$$

If W is the work done when a constant force, $\vec{F}$, of magnitude F points in the direction of the displacement, $\vec{S}$, ($\theta = 0$, $\cos 0=1$), then

$$P_{av} = \frac{\vec{F}\cdot\vec{S}}{t} = \frac{FS}{t} = F\frac{S}{t} = F\vartheta_{av} \qquad 7.5$$

Here P_{av} is the average power and ϑ_{av} is the average speed.

Work done by a variable force

The work done by a variable force cannot be calculated by the direct use of the formula. $\vec{F}\cdot\vec{S}$ for the entire displacement $\vec{S}$ because the force is continuously changing with the displacement. We consider here a simple case in which the force is changing only in magnitude. We assume that the force is pointing in the positive direction of x-axis and the body is constrained to move in that direction.

We can obtain an approximate value of the work done by a variable force in the following way.

We divide the total displacement into a large number of small intervals each of equal width Δx. A constant force is supposed to act throughout an interval. This constant force is taken to

be the force which really acts at the beginning of each of intervals. Obviously such a chosen constant force is different for different intervals. The work done by the force F(x) acting in the interval Δx constructed at x is

$$\Delta W = F(x)\,\Delta x$$

We calculate the work done by the force for each interval in the above manner. The total work done in displacing the body from x_1 to x_2 is approximately equal to the sum of all the terms like $F(x)\,\Delta x$

$$\therefore \quad W = \sum_{j=1}^{N} F(x_j)\,\Delta x_j \qquad 7.6$$

Where $F(x_j)$ is the force acting in the jth interval of width $\Delta x_j = \Delta x$, and N is the total number of intervals.

Although the width of each interval is small but still finite. Hence the force acting at the beginning of each interval is not the force which is actually acting at each point of the interval. It can be taken only approximately for the whole interval. The result so obtained is, thus, only approximate. In order to obtain a better result for the work done we divide the total displacement into a larger number of intervals so as to make the width of each interval smaller. As a matter of fact the accuracy of the result depends on the extent to which the width of each interval is made small. A far better result approaching the exact one is obtained by dividing the displacement into an infinitely large number of intervals so as to make the width of each interval infinitesimally small. An exact result is obtained if N tends to infinity and so Δx tends to zero. Thus we get

$$W = \lim_{N \to \infty} \sum_{j=1}^{N} F(x_j)\,\Delta x_j$$

The above summation is equivalent to the integral

$$\int_{x_1}^{x_2} F(x)\,dx$$

$$W = \lim_{N \to \infty} \sum_{J=1}^{N} F(x_j)\, \Delta x_j = \int_{x_1}^{x_2} F(x)\, dx$$

We now give a graphical interpretation to the expression $\sum_{J=1}^{N} F(x_j)\, \Delta x_j$ as follows

The force F(x) is plotted versus x so as to obtain a curve AB as shown in Fig. 7.5(a). The work done by the force while the object moves through the first interval is $F(x_1)\,\Delta x_1 = F\,(x_1)\,\Delta x$. Obviously it is nearly equal to the area of the first rectangular strip. Similarly the work done in moving the object through the second interval is $F\,(x_2)\,\Delta\, x_2 = F\,(x_2)\,\Delta x$.

It is approximately equal to the area of the second rectangular strip. The process goes on till we reach the final interval for which the work done is $F(x_N)\,\Delta x_N = F(x_N)\,\Delta x$. which is nearly equal to the area of the final strip. Hence the total work done is approximately equal to the sum of the areas of the rectangular strips Fig.7.5(a). In Fig. 7.5(b) the number of strips is greater and hence the width of each strip is smaller than that in Fig.7.5(a).

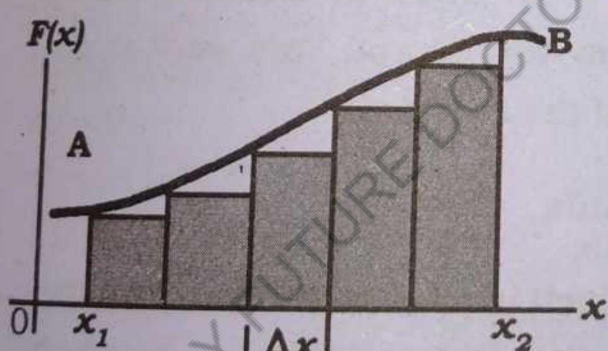


Fig. 7.5(a).

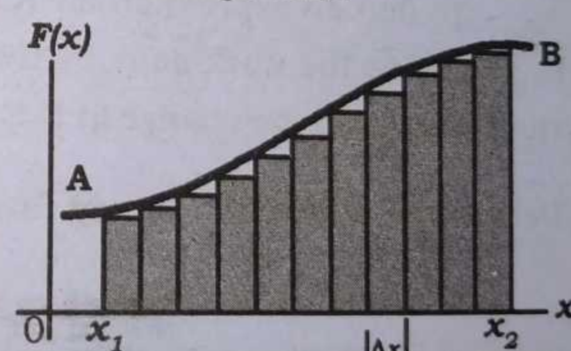


Fig. 7.5(b)

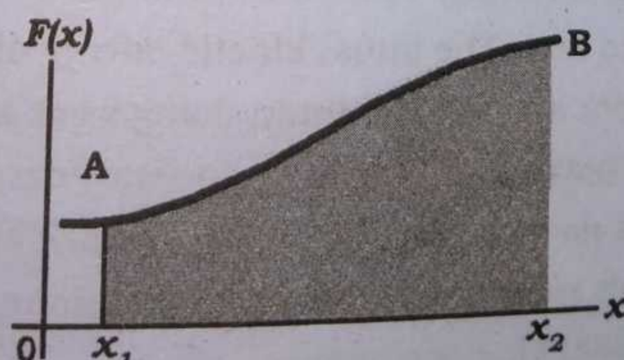


Fig. 7.5(c)

Fig: 7.5 *(a) Dividing the area into a few strips each of width Δx.*
(b) The strips are narrower and more in number.
(c) The strips are only infinitesimal in width.

Fig. 7.5(c) corresponds to the case when $N \to \infty$ so that $\Delta x \to 0$ and the work done in displacing the body from x_1 to x_2 is exactly equal to the area under the curve and bounded by the axis of x at its two limits x_1 and x_2.

7.4 ENERGY

Energy is the capacity of doing work. Energy is associated with the performance of work because the more work that is done, the greater the quantity of energy is needed. Work is always done by a force. It means that a body possessing an energy can exert force on any other body to do work. On the other hand when a work is done on a body, an equal amount of energy is stored in it.

Kinetic Energy The energy possessed by a body by virtue of its motion is called kinetic energy K.E. i.e moving ball can break a glass window, a striking hammer can drive a nail and a stone thrown upward can lift itself against the force of gravity.

To find an expression for K.E of an object in motion we have to determine the work done by the moving object. This work is obviously equal to the change in K.E of the object.

Derivation of Kinetic energy formula

Consider a body of mass m which is projected up in the gravitational field with a velocity V. After attaining a maximum height h the body comes to rest. The initial kinetic energy of the body is capable of doing work and is used up in doing work against the force of gravity. At the maximum height the kinetic energy of the body is zero because it is no more capable of doing work against the gravitational force. This means that the total work done by the body is a measure of its initial kinetic energy.

Work done by the body $= \vec{F} \cdot \vec{S} = FS\cos\theta \quad [\because \theta = 0,\ \cos\theta = 1]$

$$= FS = mgh.$$

We will eliminate h from the above equation by using the equation of motion

$$V_f^2 - V_i^2 = 2as$$

where

V_i = V = initial velocity of the body

V_f = 0 = final velocity of the body.

a = $-g$ = acceleration of the body.

S = h = maximum height attained by the body

$\therefore$

$$-V^2 = -2gh \text{ or } V^2 = 2gh$$

$$\therefore\ h = \frac{V^2}{2g}$$

Substituting this value of h in the expression for work done we get

$$\text{total work done} = mgh = (mg) \times \frac{V^2}{2g} = \frac{1}{2} mV^2$$

Hence the Kinetic energy of a body of mass m and moving with velocity V is

$$\text{K.E (kinetic energy)} = \frac{1}{2} mV^2$$

One should note that the expression for Kinetic energy is independent of the way by which it is derived.

Example

A neutron travels a distance of 12m in a time interval of 3.6×10^{-4}s. Assuming its speed was constant; find its Kinetic energy. Take 1.7×10^{-27} kg as the mass of neutron.

Solution:

$$V = \frac{S}{t} = \frac{12m}{3.6 \times 10^{-4} s} = 3.3 \times 10^4 \text{ m/s}$$

The Kinetic energy is

$$K.E = \frac{1}{2}mV^2 = \frac{1}{2}(1.7 \times 10^{-27} kg)(3.3 \times 10^{4} m/s)^2$$

$$= 9.256 \times 10^{-19} \text{Joule}$$

$$= 5.78 \text{ eV}$$

7.5 POTENTIAL ENERGY

When a body is being moved against a field of force, an energy is stored in it. This energy is called Potential Energy? For example if we compress a spring, an elastic potential energy is developed in it; this energy is stored in it because a work is done in compressing the spring against the elastic force. Similarly when an electric charge is moved against an electrostatic force, work is done on the charge. This work done is stored in it in the form of electrostatic potential energy. Similarly when a body of mass m is lifted to a height h against the gravitational force, (weight of the body), work is done on it. This work is stored in it in the form of gravitational potential energy.

In order to derive an expression for the gravitational potential energy at a height (very near to the surface of the earth) consider a ball of mass m which is taken very slowly to the height h as shown in Fig 7.6. The very slow motion is possible only when the applied force on the body by an external agency is equal in magnitude to that of the force of gravity that is

$$F_{ex} = mg$$

where F_{ex} represents the magnitude of the force $\vec{F}_{ex}$

Under this condition the kinetic energy and the acceleration of the body is zero.

Work done by the applied force $= W_{ex} = \vec{F}_{ex} \cdot \vec{S}$

$$= W_{ex} = F_{ex}h, \; [\because S = h]$$

$$= W_{ex} = mgh$$

Since an external force and the displacement are taking place in the same direction. (i.e. $\theta = 0$, $\cos\theta = 1$)

Work done by the gravitational force $(W_g) = \vec{F}g . \vec{S} = -mgh$ minus sign indicates that the force of gravity and the displacement are oppositely directed.

Comparing the two equations for work we get

$$W_g = -W_{ex} \Rightarrow W_{ex} = -W_g$$

The work done on a body by an external agency (applying an external force) against the gravitational force is stored in it in the form of potential energy and is known as the gravitational potential energy represented by U_g. Thus

$$U_g = W_{ex} = -W_g = -(-mgh)$$

$$= mgh \qquad 7.10$$

Thus potential energy of a body in the earth's gravitational field at a height h is mgh which is a positive quantity with respect to that at the surface of the earth which is supposed to be the level of an arbitrary zero potential energy.

SI unit of gravitational potential energy is Joule (J).

7.6 ABSOLUTE P.E

The above expression in Eq.7.10 for the gravitational potential energy is the relative potential energy of the body with respect to some arbitrary zero level, at which the potential energy is not actually zero. Now we want to find the P.E of a body with respect to a point at which P.E is actually zero. Obviously this point is very far away from the centre of the earth. The P.E of a body at a height h from the centre of the earth with respect to the above said point (at which the gravitational field is zero), is called the absolute P.E.

Absolute gravitational potential energy:-

In the derivation of an expression for the relative P.E (mgh).

we have assumed that throughout the displacement of the body, from the initial position to the final position, force of gravity remains constant. No doubt the above assumption, for the constancy of the gravitational field, is valid, if the height, to which the body is displaced, is not large. On the other hand, when we consider problems involving large displacements (height h) as measured from the surface of the earth, e.g in space flights we cannot take the gravitational force as constant. In fact, it decreases with the increase of height. Hence to calculate the work done (which is a measure of P.E) against the force of gravity, the simple formula $\vec{F}.\vec{S}$ can not be applied.

To overcome this difficulty we divide the entire displacement into a large number of small displacement intervals and applying Newton's Law of Gravitation.

In Fig. 7.6, a point B is situated at a very large distance from the surface of the earth, in the gravitational field. We want to calculate the work done against force of gravity, in taking a body of

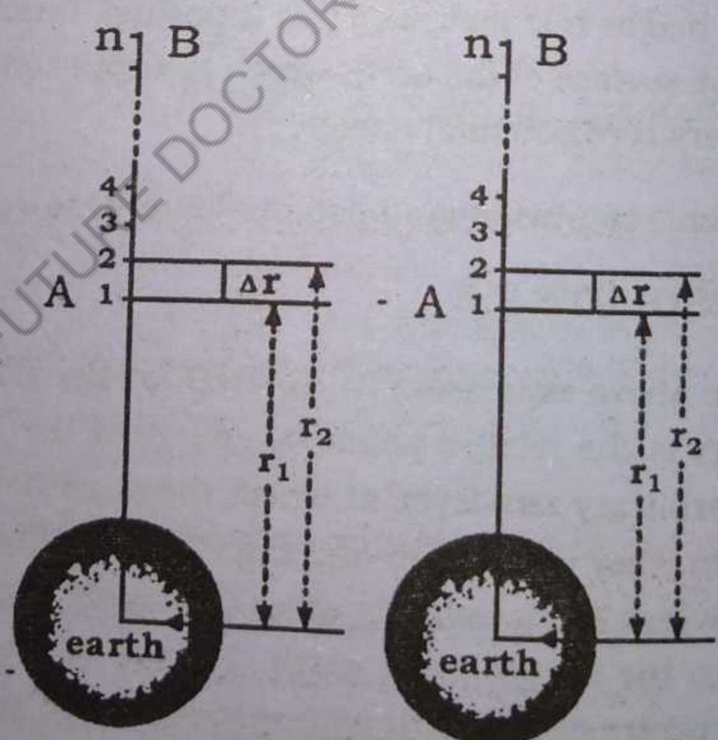


Fig : 7.6

mass m from an initial position A (or 1) to the final position B (or n). We divide the distance between A & B into a large number say,n of intervals of equal small width Δr each.

Since Δr is small, the force of gravity throughout this interval may be assumed to be constant. This value of constant force may be taken as the average of the forces acting at the two ends of an interval.

The magnitude F_1 of the force $\vec{F}$ acting at the point 1 (first end of the first interval) is given by

$$F_1 = \frac{Gm\,Me}{r_1^{\,2}}$$

Here M_e is the mass of earth, G is universal gravitational constant and r_1 is the distance of the point 1 from the centre of the earth.

Similarly the magnitude F_2 of the force $\vec{F_2}$, acting at point 2 is given by

$$F_2 = \frac{Gm\,M_e}{r_2^{\,2}}$$

The average froce acting throughout the first interval

$$F = \frac{F_1 + F_2}{2}$$

where F represents the magnitude of the average force $\vec{F}$ therefore

$$F = \frac{Gm\,M_e}{2}\left[\frac{1}{r_1^{\,2}} + \frac{1}{r_2^{\,2}}\right]$$

$$= \frac{Gm\,M_e}{2}\left[\frac{r_2^{\,2} + r_1^{\,2}}{r_1^{\,2}\ r_2^{\,2}}\right]$$

$$= \frac{Gm\,M_e}{2}\left[\frac{(r_1 + \Delta r)^2 + r_1^{\,2}}{r_1^{\,2}\ r_2^{\,2}}\right] ; \left[\because \begin{array}{c} r_2 = r_1 + \Delta r \\ \text{(from figure)} \end{array}\right]$$

$$= \frac{Gm\,M_e}{2}\left[\frac{r_1^{\,2} + 2r_1\,\Delta r + (\Delta r)^2 + r_1^{\,2}}{r_1^{\,2}\ r_2^{\,2}}\right]$$

as Δr is very small, $(\Delta r)^2$ is negligibly small

$$= \frac{Gm\,M_e}{2}\left[\frac{2r_1^2 + 2r_1\Delta r}{r_1^2\;r_2^2}\right] = \frac{Gm\,M_e}{2}\;\frac{2r_1(r_1+\Delta r)}{r_1^2\;r_2^2}$$

$$F = \frac{Gm\,M_e}{r_1\;r_2}$$

The work done in lifting the body from point 1 (position A) to point 2, by an applied force, which is equal and opposite to the average gravitational force, is given by

$$W_{12} = \vec{F} \cdot \vec{\Delta r}$$

Since the applied force $\vec{F}$ and displacement $\vec{\Delta r}$ are in the same direction.

$$W_{12} = F\,\Delta r \;:\; \because F = |\vec{F}|,\; \Delta r = |\vec{\Delta r}|$$

substituting for F, Δr in the above equation, we get

$$W_{12} = (GM_e\,m)\,\frac{r_2 - r_1}{r_1\;r_2} = GM_e\,m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Similarly the work done in lifting the body from point 2 to 3, 3 to 4, ... and so on

$$W_{23} = GM_em\left(\frac{1}{r_2} - \frac{1}{r_3}\right)$$

$$W_{(n-1)n} = GM_em\left(\frac{1}{r_{n-1}} - \frac{1}{r_n}\right)$$

Hence the total work done by the applied force in lifting the body from initial position A to final position B, we get

$$W = W_{12} + W_{23} + \ldots\ldots\ldots + W_{(n-1)(n)}$$

$$= GM_em\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + GM_em\left(\frac{1}{r_2} - \frac{1}{r_3}\right) + \ldots + GM_em\left(\frac{1}{r_{n-1}} - \frac{1}{r_n}\right)$$

$$W = GM_em\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_3} - \frac{1}{r_4} + \ldots + \frac{1}{r_{n-1}} - \frac{1}{r_n}\right]$$

$$W = GM_em\left(\frac{1}{r_1} - \frac{1}{r_n}\right)$$

This is the P.E represented by U of the body at the point B with respect to the point A. Hence the potential energy of the body at the point A with respect to that at the point B is $\Delta U = -W$

$$\Delta U = -GM_em\left(\frac{1}{r_1} - \frac{1}{r_n}\right)$$

$$\Delta U = GM_em\left(\frac{1}{r_n} - \frac{1}{r_1}\right)$$

when the point B lies at an infinite distance i.e. $r_n = \infty$, the p.E. at that point is zero (this point becomes reference point) then $U = \Delta U$.

$$U = (P.E)_{abs} = -\frac{GM_em}{r_1} \qquad 7.11\ (a)$$

assigning an arbitrary value to r_1 (i.e., $r_1 = r$), the Eq. 17.11 (a) can be

$$U = (P.E)_{abs} = -\frac{GM_em}{r} \qquad 7.11\ (b)$$

Therefore the absolute P.E of a body of mass m lying at the surface of the earth is given by

$$(P.E)_{abs} = -\frac{GM_em}{R_E} \qquad (7.12)$$

Where R_E is the radius of earth

The minus sign indicates that the potential energy is negative at any finite distance that is the potential energy is zero at infinity and decreases as the separation distance decreases. This is due to the fact that the gravitational force acting on the particle by the earth is attractive. As the particle moves in from infinity, the work W_∞ is positive which means U is negative.

An approximate value of the absolute potential energy at a height h ($h << R_E$) above the surface of the earth can be obtained from the Eq. 7.12.

$$(P.E)_{abs} = -\frac{GM_em}{R_E} = -\frac{GM_em}{R_E + h} = -\frac{GM_em}{R_E(1 + h/R_E)}$$

$$= -\frac{GM_em}{R_E}\left(1 + \frac{h}{R_E}\right)^{-1}$$

The expression under brackets given by binomial expansion as

$$\left(1 + \frac{h}{R_E}\right)^{-1} = 1 - \frac{h}{R_E} + \left(\frac{h}{R_E}\right)^2 - \left(\frac{h}{R_E}\right)^3 + \text{--------}$$

$$\approx 1 - \frac{h}{R_E}$$

Where we have neglected higher order terms and there fore,

$$(P.E)_{abs} = -\frac{GM_em}{R_E}\left(1 - \frac{h}{R_E}\right) \qquad (7.13)$$

7.7 INTERCONVERSION OF P.E AND K.E (FREELY FALLING BODY)

Consider a body of mass m placed at a point P which is at a height h measured from the surface of the earth. The body possesses a gravitational potential energy, P.E; equal to mgh with respect to point O lying on the surface of the earth. We assume that

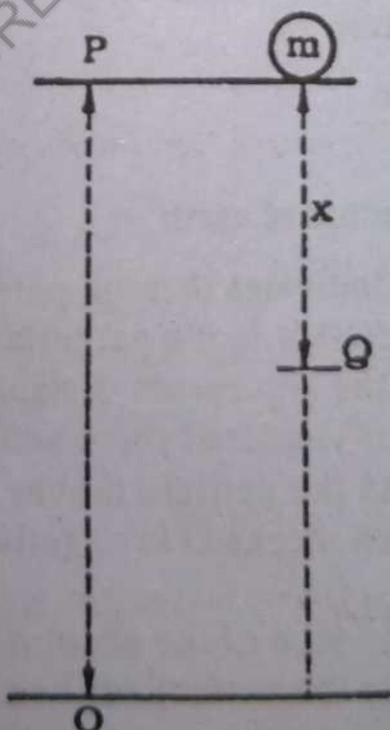


Fig: 7.7

the surface of the earth is a level of zero P.E. Suppose the body is falling freely under the action of gravity. Consider its position Q at a distance x below the point p during downward motion. Obviously the P.E of the body at this point is mg (h – x). This value of P.E is less than mg h i.e. mg (h-x) < mgh. This means that the body has lost P.E by an amount mg x. Where has P.E gone? The answer to this question is as follows.

The body at P has zero K.E because it is at rest. During its downward motion, its velocity increases and so there is an increase in K.E. We assume that there is no force of friction involved during the motion of the body thus the loss of P.E must be equal to the gain in K.E i.e. the P.E is being converted into K.E.

Eventually when the body reaches just above the point O, its P.E is nearly zero i.e whole of its P.E is converted into K.E. It means that at every point, during the fall of the body, assuming that there is no friction, we have

Loss of P.E = Gain in K.E

In practice there is always a force of friction f, say opposing the downward motion of the body. Here a fraction of the P.E is used up in doing work against the force of friction. Thus a modified form of the above equation is

Loss of P.E = Gain in K.E + work done against friction

or Gain in K.E = Loss of P.E - work done against friction

$$= mg\,x - fx \qquad (7.14)$$

Here f is average frictional force and if x is replaced by h

$$\therefore \quad \text{Gain in K.E} = mgh - fh \qquad (7.15)$$

An equation like (7.15) is called work energy equation. This equation is very useful in solving problems.

7.8 LAW OF CONSERVATION OF ENERGY

The law states that energy can neither be created nor can it be

destroyed. It can only be transformed from one form to another. A loss in one form of energy is accompanied by an equal increase in the other forms of energy. The total energy remains constant.

Let us elaborate the above statement. Energy cannot be created means one cannot produce energy by expending nothing. We get energy only by expending something appropriate. Similarly we can not destroy energy. We get something equivalent in return if we annihilate it. *Pair production* is a good example of annihilation of energy. On the other hand in nuclear reactions (*fission and fusion*) energy is created at the cost of mass. If m is the mass annihilated, then according to Einstein's famous mass-energy relation, the energy produced is

$$E = mc^2$$

where c is the velocity of light in vacuum.

The law of conservation of energy is universally accepted because there is not a single example in which it is contradicted. Although the law seems to be very simple but its implication is very important. For example if one says that a machine can be invented, which works without expenditure of energy or fuel, we will immediately discard his statement because in every real machine there is always a force of friction which must be overcome by expenditure of energy.

The law of conservation of mechanical energy (Kinetic and potential) is a particular case of the general law of conservation of energy. Here interconversion of potential and Kinetic energies takes place.

With reference to the problem of a freely falling body as discussed in section 7.5, the relative potential and the Kinetic energies at the point 'P' are given as

$$P.E = mgh, \quad K.E = 0$$

$$\therefore \quad P.E. + K.E. = mgh$$

We now calculate the kinetic energy at 'O'. which is given by

$$K.E. = \frac{1}{2} m V^2$$

where V is the velocity with which the body reaches the point O. Using the equation of motion we obtain the velocity as

$$V_f^2 - V_i^2 = 2gh$$

Where V_i = initial velocity (at P) = 0

V_f = final velocity (at O) = V

$$\therefore \quad V^2 = 2\,gh$$

Hence $K.E = \frac{1}{2} mV^2 = \frac{1}{2}\, m \times 2gh = mgh.$

The potential energy at 'O' is taken arbitrarily equal to zero with respect to which the potential energy at the point P is mgh.

$$P.E + K.E = mgh$$

We now calculate the potential and Kinetic energy at any point Q at a distance x below the point P.

$$P.E' = mg\,(h - x) \quad \text{and} \quad K.E. = \frac{1}{2} m V^2$$

The velocity V is calculated in similar manner as before h. is replaced by 'x'

$$K.E = mgx$$

$$\therefore P.E + K.E = mg\,(h\text{-}x) + mgx$$

$$= mgh \qquad 7.16$$

This shows that the sum of kinetic energy and the potential energy (total energy) is always constant provided there is no force of friction involved during the motion of the body.

If we measure the velocity of the body just before it touches the ground and calculate the corresponding kinetic energy, we find that the measured, kinetic energy (say, $\frac{1}{2}mV^2$) at the bottom is not equal to the potential energy mgh at the top, that is, $\frac{1}{2}mV^2 < mgh$. One may think that perhaps the conservation law of energy is violated. This statement is not correct. The apparent violation is due to the fact that we have not taken into account the force of friction which acts on the falling body. Work is done against the force of friction for which energy is required. This energy comes from the initial potential energy of the body. When this work is added to the measured kinetic energy, the sum is always equal to the initial potential energy of the body at the top.

Examples of conservation of energy from every day life:

(i) When we switch on our electric bulbs, their filaments are heated up and begin to emit light. In switching on the bulb we supply electrical energy to it. It is converted into heat and light energies. Here one form of energy (electrical energy) transforms into another forms (heat and light) of energies. But the electrical energy supplied is equal to the sum of the heat and light energies. In this example the energy is neither created nor destroyed.

(ii) Fossil fuels e.g. coal and petrol are stores of chemical energy. When they burn, chemical energy is converted into heat energy, that is,

chemical energy = Heat energy + losses.

(iii) The heat energy present in the steam of a boiler develops such a large pressure that it drives a steam engine. Here heat energy is converted into kinetic energy (mechanical energy), that is,

Heat energy = Mechanical energy + losses.

(iv) In rubbing our hands we do mechanical work which produces an equal amount of heat energy, that is,

Mechanical energy = Heat energy + Losses.

7.9 VARIOUS SOURCES OF ENERGY

So far we have discussed the KE & P.E. There are many other forms of energy extracted from different sources e.g. Wind energy, Hydro electricity, Chemical energy, Fossil-fuel energy, Nuclear energy, Geothermal energy, Solar energy ,Tidal energy etc. We give a brief description of each as follows.

(i) Wind Energy (Wind Power)

The source of this energy is the wind. This energy is used in running flour mills. In Karachi near Suhrab Goth you can see a wind mill for drawing underground water.

(ii) Hydro electricity (Water Power)

Mangla dam, Tarbela dam and other dams in our country are used to produce electrical energy. Their prime function, is to retain river water so that it can be shuttled off to a water turbine that drives an electrical generator. The principle involves a way of supplying power to a generator other than by a steam turbine.

(iii) Fossil Fuel

Fossil fuels are remnants of plants and animals which died millions of years ago. Depending on the conditions of formation, the fuel can be liquid (crude oil), gaseous (natural gas), or solid (coal,peat,lignite). Coal is being used by man since long as a source of energy. In present age the main source of energy is gasoline. Fossil fuel is used for running machines for driving engines etc.

(iv) - Nuclear Energy

The nuclear energy is produced due to the fission of a heavy

nucleus. If fission reaction occurs in a controlled manner (in a reactor), the nuclear energy is used to produce electrical power. A nuclear reactor is working in Karachi to generate electrical power. The energy thus produced is more economical and non polluting. If fission reaction is uncontrolled the enormous energy produced in the form of heat causes heavy destruction. The destruction of Japan due to it is a tragic example.

Fusion reaction, if uncontrolled can cause much more destruction than that caused by fission reaction. On the other hand controlled fusion reaction (CFR) may generate enormous amount of energy for useful purpose for which the scientists are working all over the world

(v) Geothermal Energy

Geothermal energy is the earth's natural heat. Heat, in fact, conducted out from the interior of the surface of planet (earth) at a rate of approximately 1.5 μ cal/cm^2-s and over a time interval of a year, this flux to the entire surface 1020 cal. D.E. White and D.L. William estimated that the heat stored in rock beneath the USA to a depth of 10km is of the order of 8×10^{24} cal. Of course, a lot of this heat is not usable. Practically, heat must be concentrated in geothermal reservoirs where it is to be exploitable. It is interesting to observe, however, that in the upper 10 km (when the temperature exceeds 100°c) the total stored geothermal energy exceeds, by order of magnitude, all thermal energy available in all nuclear and fossil fuel sources.

The man is aware of hot springs, water of which have been heated by hot rocks. Boric acid and other chemicals are extracted from the steam jets in Italy as early as 1700 A.D. Larderello Company produced electricity in 1904 using natural steam for the generation of power.

(vi) Solar Energy

Solar energy is by far our most available energy source. Our lives absolutely depend on it for food production and we call on it for a multitude of things ranging from sun tanning to clothes drying. Solar energy could make a major impact on our energy economy (1) providing space heating, space cooling, and hot water building (2) providing clean fuels and (3) generating electricity by solar cells.

(vii) Tidal Energy

The thought of harnessing the enormous energy content of both the ocean and tides have pervaded the minds of human being for centuries. The tides have their origin in the gravitational force exerted on the earth by the moon and the sun. Water-powered mills operating from tidal motion were used in New England in the 18th century. Sewage pumps functioned in Germany and London by using tidal power. These systems were replaced by the more economical and convenient electric motors.

Although no source exists that renders less environmental damage, tidal energy is difficult to harness and marginally economical.

The fossil fuel is used as the main source of energy in Pakistan. It requires a huge amount of foreign exchange to import it. Due to its burning environmental damage is done on a very high scale. The hydro electric generation is also limited and also costly. For our present and future needs we must provide indigenous atomic reactor to generate electrical power. Along with solar energy should be exploited to a greater extent. Solar energy is ideal source of energy to get rid of pollution. Solar energy is available in most of the parts of Pakistan throughout the year.

Solved examples.

(1) A person pushes a toy car, initially at rest, towards a child by exerting a constant horizontal force $\vec{F}$ of magnitude 5N through a distance of 1 m. (a) How much work is done on the

car? (b) What is its final kinetic energy? (c) If the car has a mass of 0.1 kg. What is its final speed? (Assume the absence of frictional forces).

Solution

(a) The force the person exerts on the car is parallel to the displacement, so the work done on the car by the person is

$$W = \vec{F}.\vec{S} = FS = (5N)(1m) = 5J$$

(b) The initial kinetic energy K_o is zero, so the final kinetic energy of the car is

$$K = K_o + W = 0 + 5J = 5J.$$

(c) The final kinetic energy is

$$K = \frac{1}{2} m\vartheta^2$$

$$\text{or } \vartheta = \sqrt{\frac{2k}{m}} = \sqrt{\frac{(2)(5J)}{(0.1kg)}} = 10m/s$$

(2) A 70 kg man runs up a hill through a height of 3 m in 2 seconds. (a) How much work does he do against gravitational forces ? (b) What is his average power output?

Solution

(a) The work done, ΔW, is equal to change in his potential energy, mgh. Thus

$$\Delta W = mgh = (70kg)(9.8\ m/s^2)(3m)$$

$$\Delta W = 2060J$$

(b) His average power is the work done divided by the time.

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{2060J}{2s} = 1030 \text{ watts}$$

(3) What is the change in gravitational potential energy when a 7000 N elevator moves from street level to the top of a building 300 m above the street level?

Solution

The gravitational potential energy of the system (elevator+earth) is U = mgh.

$\because \quad \Delta U = U_2 - U_1 = mg(h_2 - h_1)$

But mg =7000 N and $h_2 - h_1 = 300m$

$\because \quad \Delta U = 7000\ N \times (300m)$

$= 2100000\ J$

$= 2.1 \times 10^6\ J$

Problems

1 Calculate the work done by a force F specified by

$\vec{F} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

in displacing a body from position B to position A along a straight path. The position vectors A & B are respectively given as

$\vec{r}_A = 2\hat{i} + 5\hat{j} - 2\hat{k}$

and $\vec{r}_B = 7\hat{i} + 3\hat{j} - 5\hat{k}$

(Ans: 8J)

2. A 2000 kg car travelling at 20 m/s comes to rest on a level ground in a distance of 100 m. How large is the average frictional force tending to stop it?

(Ans. 4000 N)

3. A 100-kg man is in a car travelling at 20 m/s. (a) Find his kinetic energy. (b) The car strikes a concrete wall and comes to

rest after the front of the car has collapsed 1 m. The man is wearing a seat belt and harness. What is the average force exerted by the belt and harness during the crash?

(Ans. (a) 20,000 J (b) 20,000 N)

4. When an object is thrown upward, it rises to a height 'h'. How high is the object, in terms of h, when it has lost one-third of its original kinetic energy?

(Ans. h/3)

5. A pump is needed to lift water through a height of 2.5 m at the rate of 500 g/min. What must the minimum horse power of the pump be?

(Ans. 2.74×10^{-4} hp).

6. A horse pulls a cart horizontally with a force of 40 lb at an angle of 30° above the horizontal and moves along at a speed of 6.0 miles/hr. (a) How much work does the horse do in 10 minutes ? (b) What is the power out put of the horse ?

(Ans. 1.8×10^{5} ft. lb (b) 0.55 hp).

7. A body of mass 'm' accelerates uniformly from rest to a speed V_f in time t_f. Show that the work done on the body as a function of time 't', in terms of V_f and t_f is

$$\frac{1}{2} m \frac{V_f^2}{t_f^2} t^2$$

8. A rocket of mass 0.200 kg is launched from rest. It reaches a point p lying at a height 30.0 m above the surface of the earth from the starting point. In the process + 425 J of work is done on the rocket by the burning chemical propellant. Ignoring air-resistance and the amount of mass lost due to the burning propellant, find the speed v_f of the rocket at the point P.

(Ans. 60.5m/s)

Chapter: 8

Wave Motion and Sound

8.1 VIBRATORY MOTION

In our surrounding we come across many things which undergo oscillatory or vibratory motion. Some examples are the motion of a pendulum, Prongs of a tuning fork when struck sitar's string when plucked, etc. A weight attached to a stretched spring, once it is released, starts oscillating. The atoms in a solid possess vibratory motion. Similarly, atoms in molecules also vibrate relative to each other. The electrons in a radiating or receiving antenna are in rapid oscillations. An understanding of vibrational motion is essential for the discussion of wave phenomena

8.2 MOTION UNDER ELASTIC RESTORING FORCE (HOOKE'S LAW)

An important type of motion occurs when the force acting on a body is directly proportional to the displacement of the body measured from its equilibrium position. Since this force acts only toward the equilibrium position, the result is a back-and-forth motion called simple harmonic motion. Consider a body of mass m, attached to a horizontal helical spring Fig.8.1. The whole system is placed on a horizontal, smooth surface. If the spring is stretched or compressed, a small distance from its equilibrium position, and then released, the spring will exert a force on the body given by

$$F_s = -kx \qquad 8.1$$

Where x is the displacement of the body from its equilibrium position and k is a positive constant, known as the force constant of the spring. The above equation is the mathematical statement of

what is known as Hooke's law. The negative sign indicates that the force exerted by the spring on the body is always directed opposite to the displacement. For example, when x is greater than zero as in Fig.8.1 (a), the spring force is to the left that is negative. When x is less than zero as in Fig.8.1(c), the spring force is to the right that is positive. No doubt, when x = 0, as in Fig.8.1(b), the spring is neither stretched nor compressed and Fs = 0. As spring force always tends to restore than original condition of the spring, it is sometimes called a restoring force or more correctly elastic restoring force.

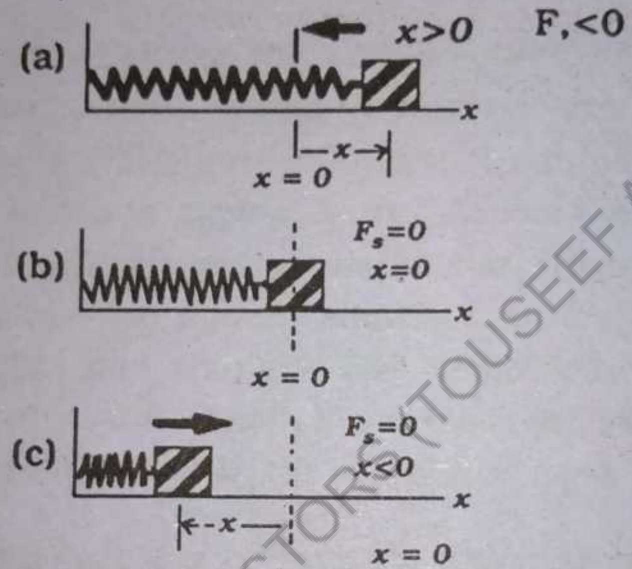


Fig. 8.1 The force of spring on a mass varies with the displacement of the mass from the equilibrium position x = 0.
(a) When x is positive (stretched spring, the spring force is to the left
(b) When x is zero (unstretched spring), the spring force is zero.
(c) When x is negative (compressed spring), the spring force is to the right.

Once the mass is displaced to some distance x_m (within elastic limit) from equilibrium position and then released, it will move from the position $-x_m$ through 0 to $+ x_m$ position. This system will continue to oscillate back and forth about its equilibrium position.

During the oscillatory motion between two extreme positions when the body passes through the equilibrium position, the total energy is kinetic. The potential energy is zero because the spring is in the unstretched condition. As the body moves to the right or left from the equilibrium position its potential energy begins to increase at the cost of kinetic energy. When the body is at the extreme position, it stops there momentarily and kinetic energy is

zero. The total energy of the system is potential energy. Since the spring tries constantly to regain its original condition, the restoring force comes into play and the body begins to move towards the equilibrium position. During this motion from extreme position to the equilibrium position, the kinetic energy increases at the cost of potential energy till the total energy is kinetic, at the equilibrium, the potential energy being zero. Due to inertia, the body then moves to the extreme position from the equilibrium position and the process is repeated for infinitely long time provided there is no loss of energy due to any internal or external factors. Using Newton's second law of motion, F = ma, the restoring force of the spring is given as

$$F_s = ma = -kx$$

$$\text{or} \quad a = -\frac{k}{m}x \tag{8.2}$$

$$= -\text{ constant} \times x$$

$$\text{or} \quad a \propto -x$$

Where the quantity (k/m) represents the constant of proportionality. Minus sign shows that the acceleration is always directed towards the equilibrium position. This back and forth (oscillatory) motion in which the instantaneous acceleration is proportional to the displacement of the oscillating body is called a simple harmonic motion. It is abbreviated as SHM

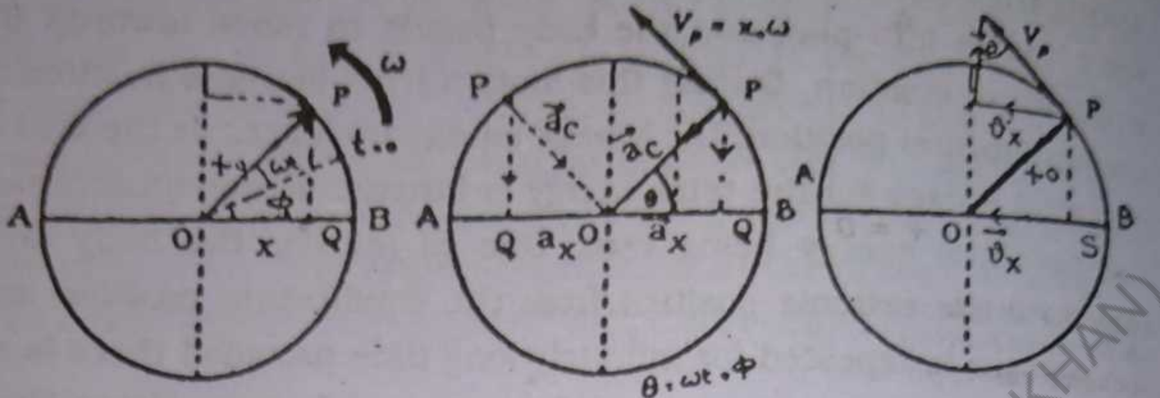
8.3 CHARACTERISTICS OF SHM

Before we derive expressions for displacement, velocity, acceleration and time period of a particle executing SHM, we compare SHM with uniform circular motion.

The connection between uniform circular motion and SHM

Many aspects of simple harmonic motion along a straight line can be better understood and visualized by showing their relationship to uniform circular motion. Consider a point mass, m, at a point P moving in a circle of radius x_o with constant angular velocity ω Fig.8.2 (a). We call this circle as our reference circle for the

motion.



(a) (b) (c)

Fig. 8.2

As the particle at point P rotates along the circumference of a circle, the projection, Q, of the particle, P, moves back and forth along the diameter AOB. At some instant of time t, the angle between OP and the x-axis is $\omega t + \phi$, when ϕ is the angle which OP makes with the x-axis at time $t = 0$. This angle ϕ is known as the initial phase angle. We take this as our reference point for measuring angular displacement. As the particle P rotates on the circle, the angle that OP makes with the x-axis changes with time and the projection of P on the x-axis, moves back and forth along the diameter of the reference circle between the two extreme positions $x = \pm x_o$.

From the right angled triangle OPQ, the displacement of point Q from the mean position is given by

$$x = x_o \cos(\omega t + \phi) \qquad 8.3$$

Notice Fig.8.2 (a) that we take x to be positive when displacement is to the right; when the displacement is to the left, $\cos(\omega t + \phi) < 0$ and the necessary negative sign is automatically supplied.

In order to give physical significance to the constant appearing in Eq 8.3, it is convenient to plot x as a function of t as shown in Figure 8.3.

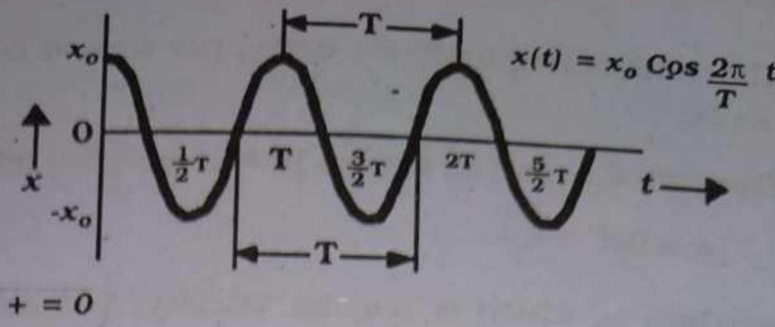


Fig. 8.3 Displacement as a function of time for a particle undergoing simple harmonic motion. There is a time interval T (The period) between any two successive corresponding point on the curve.

First we note that x_o, called the amplitude of motion, is simply the maximum displacement of the particle, in either positive or negative direction of the axis of x. The constant angle ϕ is called the phase constant or phase angle. The constant ϕ tells us what the angular displacement was at the time t = 0. The quantity $(\omega t+\phi)$ is called the phase of the motion. We also note here that the function x is periodic and repeats itself when ωt increases by 2π radians.

The time T which the particle takes to go through one full cycle of its motion is called the period or time period. This means the value of x at time t is equal to the value of x at time (t+T). We will show here that the period of the motion is given by T=2x/ω.

We know that the phase increases by 2π radians as t increases by T, then

$$\omega t + \phi + 2\pi = \omega (t + T) + \phi$$

$$\therefore \quad \omega T = 2\pi \text{ or } T = \frac{2\pi}{\omega} \qquad 8.4$$

The inverse of the period T. is known as the frequency of oscillatory motion, f. The frequency represents the number of oscillations / vibration which the particle makes in one second, i.e.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad 8.5$$

The units of frequency are cycles per second or hertz abbreviated as H_z.

Equation 8.5, can be written in a some what different form

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad 8.6$$

Thus constant ω, which is angular velocity, has units of radians per second (rad/s).

The acceleration of the point P is directed towards the centre of the circle (i.e. the centripetal acceleration) along the line PO as shown in Fig.8.2 (b). The magnitude of this acceleration is given by

$$a_c = \frac{\vartheta p^2}{x_o} = x_o \omega^2 \qquad 8.7$$

Where $\vartheta p = x_o \omega$ represents the linear speed of the point mass at P.

The acceleration of the point Q, (Q being the projection of point P) is equal to the component of the acceleration P along x-axis and is given by Fig.8.2 (b)

$$a_x = -x_o \omega^2 \cos\theta \qquad 8.8(a)$$

The minus sign is needed because the acceleration, $\vec{a}_x$, of the point Q is toward the left (along negative x-axis). When Q is left of the centre, the acceleration of P is toward the right; but since $\cos\theta$ is negative at such point the minus sign is still needed.

$$\because \quad x = +x_o \cos\theta$$

$$\therefore \quad a_x = -\omega^2 x \qquad 8.8(b)$$

Here the acceleration of point Q is proportional to its displacement and is directed towards the centre. Hence the motion of the point Q is simple harmonic. Eq 8.8(b) shows that the acceleration is maximum at the extreme positions.

$$a_{max} = \pm \omega^2 x_o$$

Comparing Eq 8.2 and Eq 8.8(b), we get

$$\omega = \sqrt{\frac{k}{m}} \qquad 8.9(a)$$

we know that

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The period of oscillation or time required for one complete trip is

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad 8.9(b)$$

The speed of the point Q is the component of the speed of the point P along diameter AOB as shown in Fig.8.2 (c)

$$\vartheta_x = \vartheta_p \sin\theta$$

$$= x_o\,\omega \sin\theta \qquad \because \vartheta_p = x_o\,\omega \qquad 8.10(a)$$

Substituting for sin θ, we get

$$\vartheta_x = x_o\omega\sqrt{1-\cos^2\theta}$$

$$\vartheta_x = \omega\sqrt{x_o^2 - x^2} \qquad 8.10\ (b)$$

$$\vartheta_x = \sqrt{\frac{k}{m}} \times \sqrt{x_o^2 - x^2} \qquad 8.10(c)$$

Equation 8.10(b) shows that the velocity is maximum at the mean position O where $x = 0$ and is equal to ωx_o ($\vartheta_{max} = \omega x_o$), and it is minimum ($\vartheta_{min} = 0$) at the extreme positions A and B.

8.4 ENERGY OF PARTICLE EXECUTING SHM

We often refer to a system that undergoes SHM as a simple harmonic oscillator. In an ideal oscillator there are no frictional

losses, so energy is conserved. At any instant of time, we have

$$E = \text{Total energy} = P.E + K.E = \text{constant}$$

To derive an expression for P.E we refer to fig. 8.1 and use Hooke's law i.e, $F = -kx$. Here the force is varying linearly with the displacement. The stretching force is zero when the displacement is zero and it is kx (only magnitude), when the displacement is x. Therefore the average force on the mass m during this displacement is $\frac{0 + kx}{2} = \frac{1}{2}kx$. Hence the work done on the mass by the force during displacement is $\frac{1}{2}kx \times x = \frac{1}{2}Kx^2$. This work done on mass is stored in it in the form of P.E. Thus the P.E of the oscillating mass at a displacement x from equilibrium position is given as

$$P.E = \frac{1}{2}kx^2 \qquad 8.11\ (a)$$

From the equation it is clear that the P.E is maximum at the two extreme positions where $x = \pm x_o$, i.e., $(P.E)_{max} = \frac{1}{2}kx_o^2$ and it is minimum at the equilibrium position i.e., where $x = o$ and $(P.E)_{min} = 0$.

Using Eq 8.10(c). the expression for the K.E is given by

$$(K.E) = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left\{\sqrt{\frac{k}{m}} \times \sqrt{x_o^2 - x^2}\right\}^2$$

$$K.E = \frac{1}{2}k(x_o^2 - x^2) \qquad 8.11\ (b)$$

From the above Equation, it is clear that K.E is maximum at the equilibrium position where $x = 0$ and the maximum value is being given as $(K.E)_{max} = \frac{1}{2}kx_o^2$. It is maximum at the two extreme positions where $x = \pm x_o$. The minimum value is being given as $(K.E)_{min} = 0$

Therefore the total energy at any instant of the oscillating mass is given by

Total energy = P.E + K.E $\qquad$ 8.11(c)

$$E = \frac{1}{2}kx^2 + \frac{1}{2}k(x_o^2 - x^2)$$

$$E = \frac{1}{2}kx_o^2 \qquad 8.11(d)$$

$$E = \text{Constant}$$

Eq. 8.11(d) shows that the total energy of the mass executing SHM is conserved throughout its displacement including mean and extreme positions

Finally, we shall now define some important terms used frequently when dealing with simple harmonic motion (SHM).

(i) Periodic motion:

A motion which repeats itself in equal interval of time is called periodic motion. If the motion is described by a periodic function f(t) with period T, then

$$f(t) = f(t \pm nT)$$

Where n = 0, 1, 2, 3.

and T is the time required to make one complete vibration/oscillation.

(ii) Frequency:

The number of vibrations made by a body in one second is called frequency. It is represented by a Greek letter (υ) read as 'nu'. It is expressed as vibrations/s or cycles/s or hertz (H_z). The reciprocal of frequency (υ) is equal to the time period (T)

8.5 EXAMPLES OF SHM

Simple pendulum

An ideal simple pendulum consists of a spherical bob sus-

pended from a light, flexible and inextensible string tied to a fixed rigid and frictionless support. When the bob is displaced from its equilibrium position, it begins to perform oscillatory motion. We will show that the bob will execute SHM, if the amplitude of oscillation is sufficiently small.

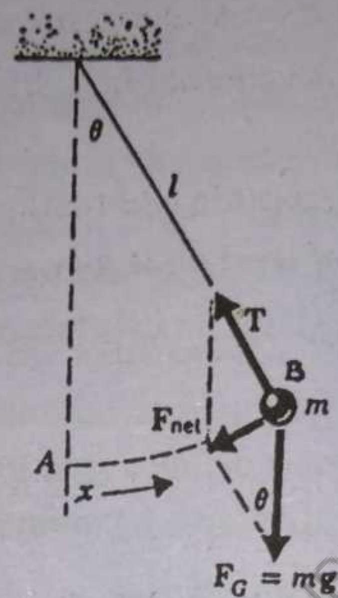


Fig. 8.4 A simple pendulum of length l. The net force on the pendulum bob is the vector sum of the gravitational force F_G and the tension in the string T.

Referring to Fig.8.4 there are two forces acting on the pendulum the vertically downward gravitational force, $\vec{F}_G = m\vec{g}$ and the tension in the suspension string $\vec{T}$. The net force acting on the bob is given as

$$\vec{F}_{net} = \vec{F}_G + \vec{T}$$

We now resolve the force $\vec{F}_G$ into two components (i) one along the length of the string of the pendulum and (ii) the other perpendicular to the string. Thus we have

$$(F_G)_{||} = mg\cos\theta$$

$$(F_G)_{\perp} = mg\sin\theta$$

Where m is the mass of the bob. Since there is no motion along the string, the net force acting in the direction of the string is zero. This is possible if the component mg cosθ balances the ten-

sion T. Hence the magnitude of the net force acting on the bob is

$$F_{net} = mg\ \sin\theta$$

This is the restoring force which is responsible for the oscillatory motion of the bob. The displacement, x, is the distance through which the bob moves along the arc (traced by the bob). Starting from A, the displacement of the hole at B is $x = l\theta$, where θ is measured in radians.

Using Newton's second law of motion and assuming $F_{net} = F$, the equation of motion of the bob is given as

$$F = ma = -\ mg\ \sin\theta$$

$$a = -\ g\ \sin\theta$$

The negative sign indicates that the force and hence acceleration are always directed towards the mean position. If θ is sufficiently small (less than about 5 degrees) then

$$\sin\theta \approx \theta$$

$$= \frac{x}{l}$$

substituting for $\sin\theta$, we write

$$a = -g\ \frac{x}{l}$$

$$a = -\ \frac{g}{l}\ x \qquad 8.12$$

$$a \propto -x$$

where (g/l) is constant of proportionality

Thus the acceleration of the pendulum is directly proportional to its displacement and is directed towards its mean position. Hence the pendulum executes SHM.

Comparing Eq. 8.8(b) and Eq 8.12 and using Eq 8.9 (a), we get

$$\omega^2 = \frac{g}{l}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}} \qquad 8.13$$

Example:

Find the time period of a simple pendulum whose length is 88.2 cm. The value of acceleration due to gravity is $9.8 m/s^2$ at the place where experiment is performed.

Solution

$$l = 88.2 \text{ cm} = 88.2 \times 10^{-2} \text{m}$$

$$g = 9.8 \text{ m/s}^2$$

$$T = ?$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$= 2 \times 3.14\sqrt{\frac{88.2 \times 10^{-2} (\text{m})}{9.8 (\text{m/s}^2)}}$$

$$= 1.885 \text{ s}$$

8.6 GENERATION OF A WAVE PULSE.

A single isolated disturbance travelling through a system is known as a wave pulse. A simple way of generating a pulse in one dimensional medium is given below.

Consider a string whose one end is tied to a fixed support while the other end is held by the hand in a stretched position. This end is given a single up and down jerk which in turn produces a hump in the string. It (hump) travels along the string forming a wave pulse as shown in the Fig.8.5. The wave pulse retains its form only if there are no frictional losses and the height of the wave pulse is not so large as to affect the tension in the string applied by the hand. During the propagation of the wave pulse, the particles

of the medium (string) move up and down in a direction perpendicular to the direction of propagation of the wave pulse.

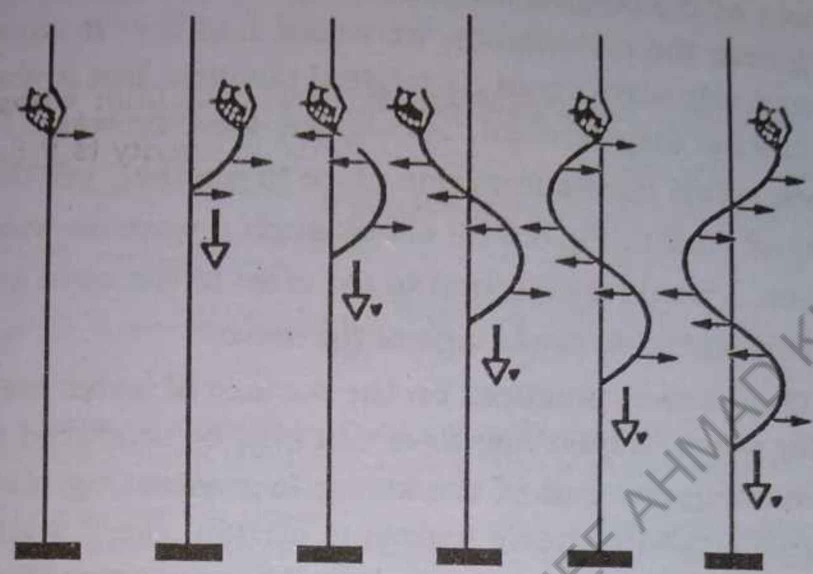


Fig. 8.5 in a transverse wave the particles of the medium (string) vibrate at right angles to the direction in which the wave itself is propagated.

The shape of the disturbed part of the string containing the wave pulse can be described at a given time mathematically by a wave function f (x) such that

$$y = f(x)$$

Here y is the vertical displacement of a particle from its equilibrium position and x is the horizontal distance of the particle from the point where the displacement y is zero. Hence x, y are the coordinates of the particle. Since the wave pulse is moving along the string, its position is changing continuously with time. Hence the location of the wave pulse depends on time. Thus the shape of the wave pulse with its location can be given by the function f (x, t) such that

$$y = f(x, t)$$

and is called wave function.

8.7 TRAVELLING WAVES

It is a common experience that disturbance is created in still

water of a pond, when a pebble is dropped into it. This disturbance produces waves on water, which move outward, finally reaching the shore of the pond. If we observe carefully the motion of a leaf floating near the disturbance, we would find that it moves up and down and side ways about its original position, but it does not undergo any net displacement. This is, the wave on water (or the disturbance which moves from one place to another, yet the water is not carried with it. Fig.8.6 (d) shows such a wave on water. In this case point A would correspond to the crest of the wave and point B would correspond to the trough of the wave.

These waves produced on the surface of water are known as travelling waves. A travelling wave can also be produced on a string by connecting one end of the string to a vibrating blade. As the blade oscillates with simple harmonic motion, every segment of the string can be treated as a simple harmonic oscillator vibrating with a frequency equal to the frequency of vibration of the blade that drives the string.

A wave is characterized by three important physical concepts, the wave length, the frequency and the speed of the wave. The wave length is defined as the distance between any two points on a wave that behave identically. For example, in the case of waves on water, the wave–length is the distance between the adjacent crests or between adjacent troughs.

Every wave travels with a certain velocity which depends on the properties of the medium through which the wave is propagating. For examples, the sound waves travel through air with a speed about 344 m/s at 20°C, whereas the speed of sound through solid is higher than this value. A special class of waves which do not need a material medium for their propagation are electromagnetic waves. they travel through a vacuum with a speed of about 3×10^8 m/s.

The frequency and the time period have already been defined in section 8.4.

8.8 TRANSVERSE WAVE

Consider a long rope under tension as shown in Fig. 8.6 (b).

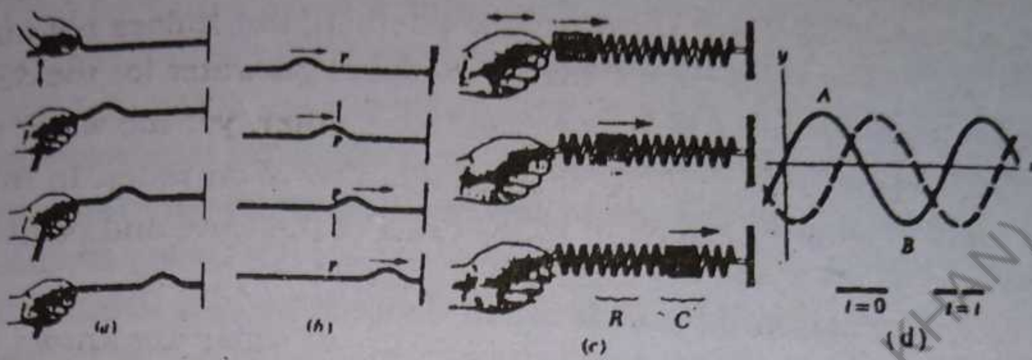


Fig. 8.6 (a) A wave pulse travelling down a stretched rope. (b) A pulse travelling on a stretched rope is a transverse wave. That is, any element P on the rope moves in a direction perpendicular to the wave motion. (c) A longitudinal pulse along a stretched spring. The disturbance of the medium (the displacement of the coils) is in the direction of the wave motion. For the starting motion described in the text, the compressed region C is followed by an extended region R. (d) A harmonic wave travelling to the right. The solid curve represents a snapshot of the wave at t = 0, and the dashed curve is a snapshot at some later time t.

one of its end is flipped to produce a wave in it. A portion of wave is produced this manner. The wave consists of a hump called pulse in the rope. This pulse travels to the right along the rope with a definite velocity. This type of disturbance is obviously a travelling wave. Fig. 8.6 (b) gives four consecutive snap shots of the travelling wave. In this case, the string is the medium through which the wave travels. We further assume that the shape of wave pulse remains unchanged while the wave travels along the rope.

It can be seen from the Fig 8.6 (c) that as the wave pulse travels along the rope, each segment of the rope, which is disturbed, moves in a direction perpendicular to the wave motion. Note that there is no motion in any part of rope in the direction of the wave propagation. A travelling wave such as this, in which the particles of the disturbed medium move perpendicular to the direction of propagation of the wave is called a transverse waves. Other examples of transverse waves are-electromagnetic waves, such as light, radio, and, television waves, etc.

8.9 ANALYTICAL TREATMENT OF TRAVELLING WAVES

We will give a simple mathematical treatment of one dimentional transvere travelling waves. Consider a wave pulse which is travelling to the right on a long stretched string with a constant speed υ as shown in Fig.8.7. The pulse travels along the x-axis and the transverse displacement of the string is measured along y-axis. Fig.8.7 (a) gives the shape and position of the pulse at time t. The shape of the pulse is represented as $Y = f(x)$. Here y is a function of x. The maximum displaement denoted by Y_{max} is called the amplitude of the wave. The distance travelled by the pulse in time t

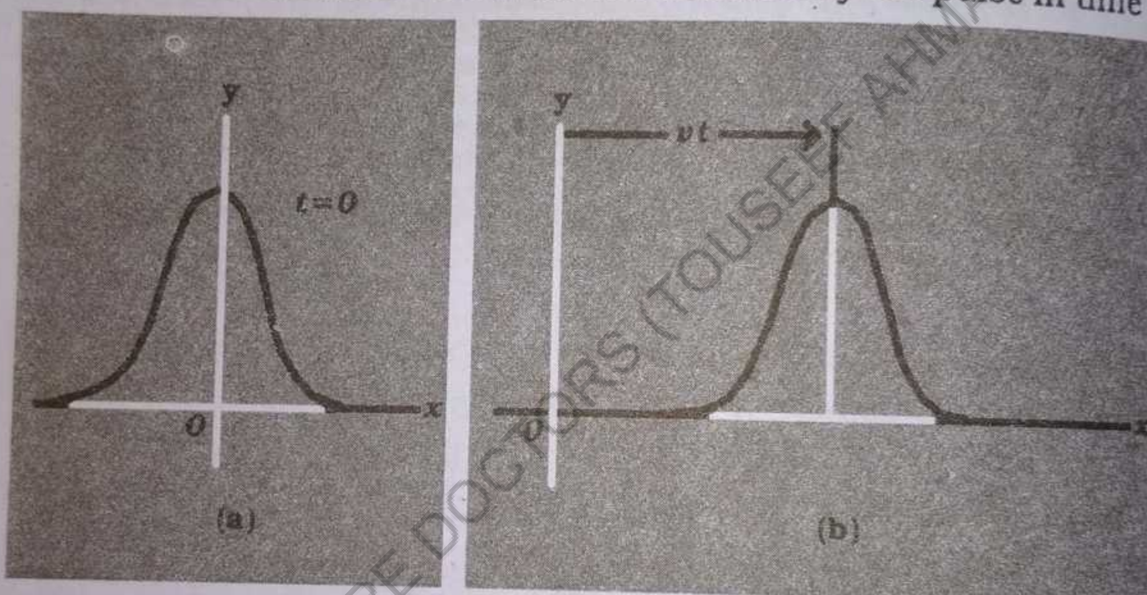


Fig. 8.7 (a) The shape of a string (in this case a pulse) at t = 0. (b) At a later time t the pulse has travelled to the right a distance x = υt.

is vt Fig. 8.7(b). Suppose the shape of the pulse does not change under this condition we can represent the displacement Y for all the later time measured in a stationary frame with the origin at O as

$$y = f(x + \upsilon t) \qquad 8.14\,(a)$$

Similarly if the wave pulse is travelling to the left, its displacement is given by

$$y = f(x - \upsilon t) \qquad 8.14\,(b)$$

Since y depends on two variables, x and t we often represent it by y (x, t), which is read, "y as a function of x and t". It is important to understand the meaning of y.

Consider a particular point P on the string, identified by a particular value of the co-ordinates x and t. As the wave passes the point P, the y coordinate of this point will increase, attain a maximum value and then decreases to zero. Therefore the wave function gives y co-ordinate of point at any time t. Furthermore, if t is fixed then the wave function y as a function of x, defines a curve representing actual shape of the pulse at this time. This is equivalent to a "snapshot" of the wave at this time.

To find the velocity of the pulse we can calculate how far its crest moves in a short time and then divide this distance by the interval. The crest of the pulse corresponds to that point for which y is maximum. In order to follow the motion of the crest, some particular value, say x_0, must be substituted for $(x - \vartheta t)$. In order to stay with the crest, we must have $x - \vartheta t = x_0$, no matter how x and t change. This gives the equation of motion of the crest. Putting t = 0 in the above equation, $x = x_0$ and at later time dt, the crest is found at a distance $x = x_0 + \vartheta dt$. Thus the distance covered by the crest in time dt is clearly $dx = x_0 + \vartheta dt - x_0 = \vartheta dt$. Thus the wave speeds which is also often referred to as the phase velocity as given by

$$\vartheta = \frac{dx}{dt}$$

The phase velocity must not be confused with the transverse velocity of a particle in the medium.

Water waves in ripple tank:

- When a stone is dropped into a smooth pond Fig.8.8 (a), a disturbance is produced. The disturbance extends over the surface

of water in the form of circles centered at the place where the stone was dropped. As a matter of fact when the stone hits the water surface, it forces the latter downwards and so produces a depression (through), with a hump (crest) all around it.

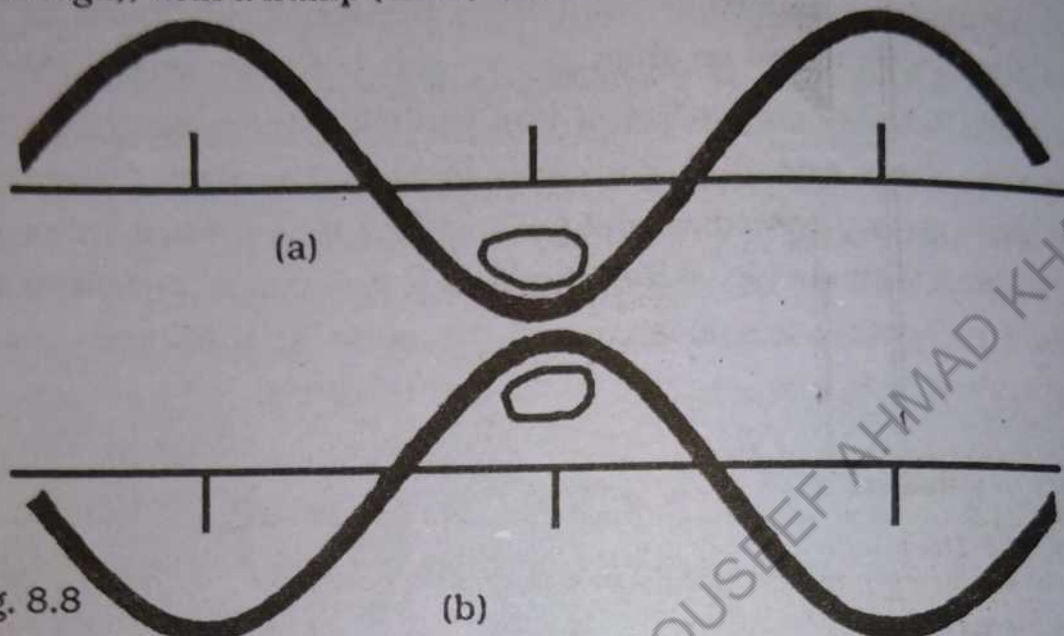


Fig. 8.8

The water regains its original position due to the difference of pressure so produced, but then instead of remaining at rest it overshoots this position because of its inertia, just as does a pendulum when pulled to one side of its equilibrium position Fig.8.8(b). Thus a circular ridge is formed. This expands into a large circle and is followed by a second circular ridge which expands and so on. The result is that the surface is soon covered with a series of circular crest. separated by circular troughs, all moving away from the centre of disturbance. Water waves can be produce by using a ripple tank show in Fig.8.9.

In its simplest form the ripple tank consists of a sheet of glass round the edges of which a beach is formed so that shallow layer of water may be held on the glass The shelving of beach helps to reduce the unwanted reflection of the waves from the sides of the tank. A bright electric lamp is placed below the tank and this forms shadow of the waves on the screen if above the tank a mirror inclined at 45 is placed so as to reflect the light on to the screen.

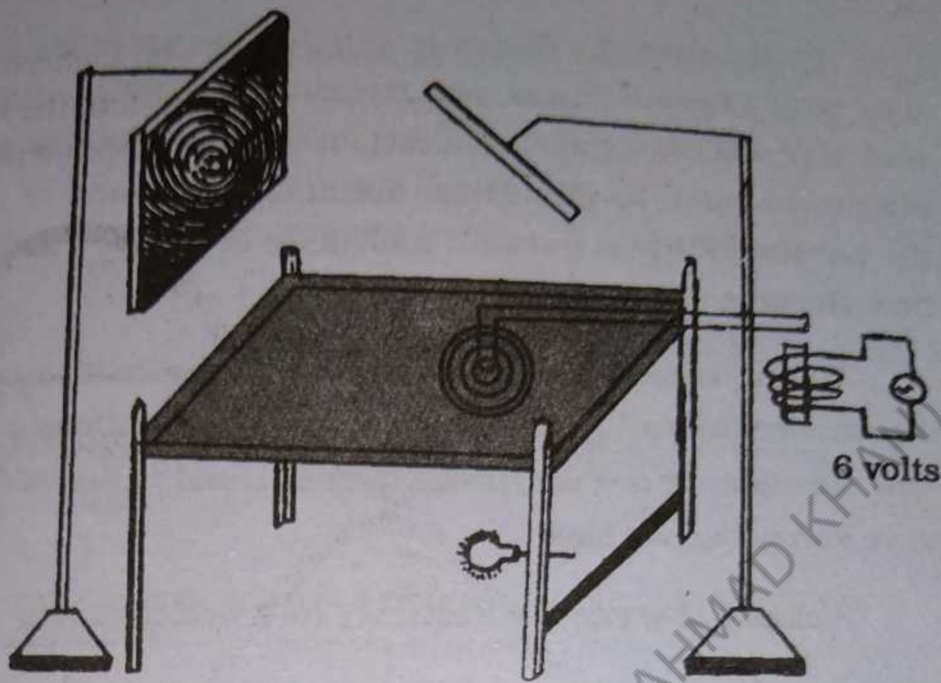


Fig. 8.9 The image of the waves is seen on the screen which is just a piece of white drawing paper placed under the ripple tank.

First we dip a finger or a pencil in the water and notice that the circular waves spread out from it whenever it moves slightly. If the finger is dipped in and out regularly at the same place, waves consisting of alternate crests and troughs follow one another at regular distances. The distance from one crest to next one is called the wave length.

If the finger is replaced by a dipper fixed to a vibrating steel strip, this will make the dippings occur at perfectly regular intervals and, further, these intervals can readily be altered by altering the length of the vibrating strip.

The state of affair set up on the water surface is called a wave motion, and three important terms are used to define it completely. These are velocity, wave length and frequency. We know that the frequency is the number of complete waves produced each second. In our example, it is the number of in and out movements made by the dipper in one second. One in and out movement of the dipper is called one complete vibration.

By the time the dipper is entering the water for the second time so as to produce a second trough, the trough produced by the first "dip" will have travelled a certain distance and this distance is clearly one wave length. Thus, during one vibration of the dipper the wavefronts travel outward a distance of one wavelength farther from the centre.

In one second the total distance a wavefront travels will be the wavelength multiplied by the number of vibrations per second. Now the distance a wave travels in one second is the velocity of the wave and hence we have

Velocity of wave(ϑ) = frequency (υ) x waelength (λ).

$$\vartheta = \upsilon\lambda \qquad (8.16)$$

knowing the frequency and wavelength, the velocity of wave can be calculated.

8.10 ENERGY IN WAVES

All mechanical waves travel through a medium, they carry energy with them and hence they are also called carriers of energy. This is easily demonstrated by having a weight on a stretched string and then sending a pulse down the string as shown in Fig. 8.10 (a). When the pulse meets the weight, the weight will be momentarily displaced as shown in Fig. 8.10 (b). Actually in this process energy is transferred to the weight since work must be done in moving it upward i.e. work is done at the cost of energy which is defined as the capability of doing work.

We describe here the state at which energy is carried along a string. We consider a sinusoidal wave (Harmonic wave) and calculate the power for this one dimensional wave.

Consider a harmonic wave travelling along a string Fig.8.10 (c). The source of the energy is some external agent at the left end

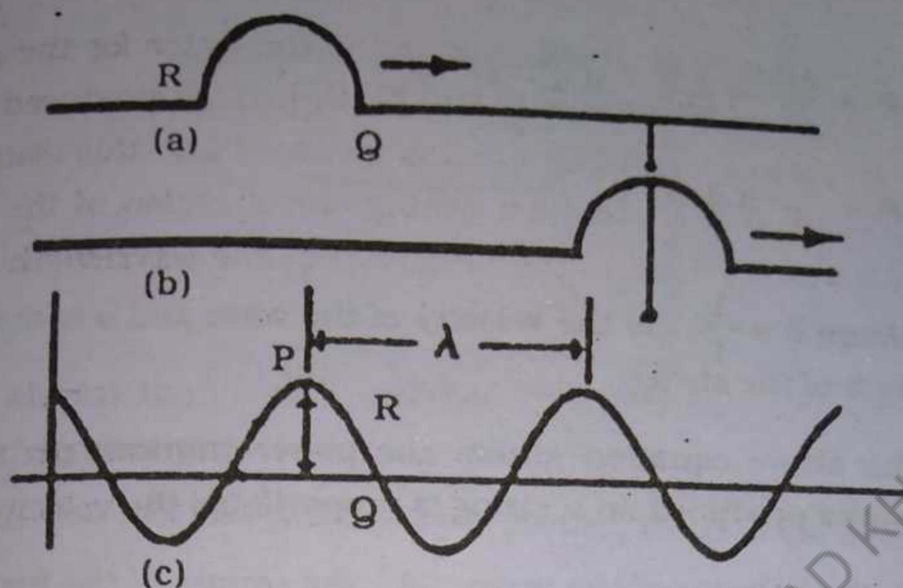


Fig. 8.10 The screen which is just a piece of white drawing paper placed under the ripple trank.

of the srting, which does work in producing the oscillation in the string. The points P,Q and R represent various segments of the string which move vertically. The wave moves a distance equal to one wavelength λ, in a time T. To determine power transmitted by the wave, we first calculate the energy contained in one wavelength and divide the result by the time T.

We know that every point on the string moves vertically up or down. Thus every segment of equal mass has the same total energy. The energy of the segment at P is entirely potential energy since the segment is momentarily stationary. The energy of the segment at Q is entirely kinetic energy and segment at R has both kinetic and potential energies. Now consider the segment Q, which has a maximum transverse velocity $(\vartheta_y)_{max}$ and mass Δm. The total energy of this segment is given by

$$\Delta E = \frac{1}{2}\Delta m(\vartheta_y)^2_{max} \qquad 8.17$$

The power, P, of the wave is obtained by dividing the above expression for ΔE by the time period T.

$$P = \frac{\Delta E}{T} = \frac{1}{2} \quad \mu \frac{\lambda}{T} \omega^2 y_o^2$$

$$P = \frac{1}{2} \mu \vartheta \omega^2 y_o^2 \qquad 8.18$$

Where $\vartheta = \frac{\lambda}{T}$ is the velocity of the wave and μ is mass per unit length of the string

The above equation shows the power transmitted by harmonic waves produced on a string is proportional to

(i) the velocity of the wave (ii) the square of the frequency and (iii) the square of the amplitude. In fact, all the harmonic waves have the following general properties.

The power transmitted by any harmonic wave is proportional to the square of the frequency and to the square of the amplitude. It should be noted here that energy is not carried by only mechanical waves (waves in a string, water waves, sound waves etc). It is also carried by electro-magnetic waves like light waves, micro waves, radio waves, thermal radiation, etc. The energy carried by electromagnetic waves also depends directly upon their frequencies. This means high frequency waves have more energy than low frequency waves. For example, energy carried by ultra violet light is much greater than those carried by infra-red light because the frequencies of ultra-violet light are much greater than those of infra-red light.

8.11 STANDING WAVES

If one end of a string, preferably a thick rubber, is fixed to a wall and the other end is held in the hand, wave can be set up along the string when streched and flicked upward. This pulse which is in the form of a crest is reflected at the fixed end and passes back along the string. The crest is reversed on reflection and becomes a trough as shown in Fig.8.11.

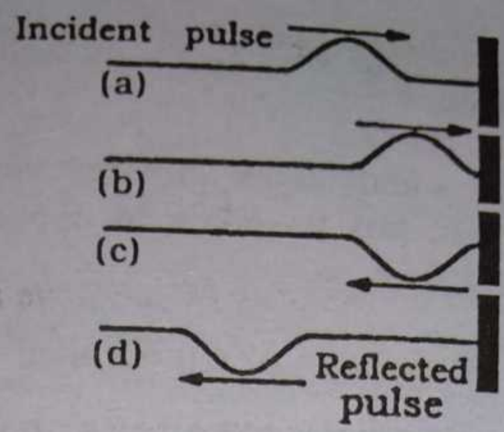


Fig. 8.11 The reflection of a travelling wave at the fixed end of a stretched string. Note that the reflected pulse is inverted, but its shape remains the same.

However, when the string is tightly stretched between two fixed supports and then plucked, the crest extends the whole distance between the supports. This distance is clearly half the wavelength of the transverse wave developed in the string as shown in Fig.8.12. At each end, the wave suffers a phase change. The crest W_1 on reflection at Q becomes a trough W_2 and the trough W_2 becomes crest W_1, on reflection at P.

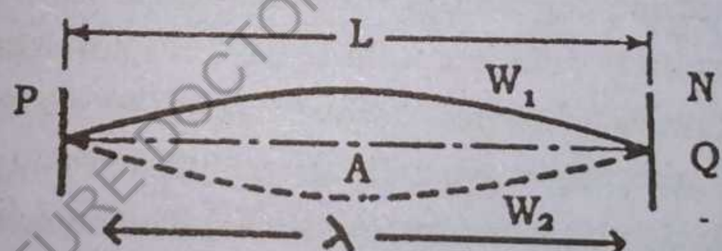


Fig. 8.12 Stationary waves set up in a stretched strings with its two clamped ends as nodes and the centre as an antinode are shown.

This simple vibration of a string consists of a transverse wave passing along the string and being reflected at each end in turn. At Q the incident and reflected waves are always equal in amplitude and opposite in phase and so the end is stationary and we get what we call a stationary wave or standing wave in the string.

The point where displacement is maximum is called antinode denoted by A. And that where the displacement is minimum (zero) is called a node denoted by N.

The incident and reflected waves will combine according to the principle of superposition.

Consider two sinusoidal waves with the same amplitude, frequency and wavelengh, but travelling in opposite directions. These waves can be written as.

$$y_1 = A_o \sin (kx - \omega t) \text{ and } y_2 = A_o \sin (kx + \omega t)$$

Where y_1 represents displacement of a wave travelling to the right taken as our incident wave and Y_2 represents displacement of a wave travelling to the left, which is our reflected wave. The resultant wave function Y is given by

$$y = y_1 + y_2 = A_o \sin (kx - \omega t) + A_o \sin (kx + \omega t)$$

Where $k = 2\pi/\lambda$ and $\omega = 2\pi f$. Using the trigonometric identity, we get

$$Y = (2A_o \sin kx) \cos \omega t \qquad 8.19$$

This expression represents the wave function of a standing wave which is entirely different from the expression which is used to describe a travelling harmonic wave. From this expression, it is obvious that a standing wave has an angular velocity ω (a constant) and an amplitude given by $2A_o \sin kx$. This means every particle of the string vibrates in SHM with same frequency ω. However the amplitude of the motion of a given particle depends on sin kx. This is in contrast to the situation involving a travelling harmonic wave, in which all particles oscillate with the same amplitude and same frequency.

As the amplitude of the standing wave for any value of x is equal to $2A_o \sin kx$ therefore the maximum amplitude has the value $2A_o$. It happens when $\sin kx = \pm 1$, i.e.

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \qquad 8.20\text{(a)}$$

Since $k = \frac{2\pi}{\lambda}$ the maximum amplitude points called anti-nodes · (also mentioned earlier), given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots, \frac{n\lambda}{4} \qquad 8.20\,(b)$$

Where n = 1, 3, 5 ... It is to be noted that anti-nodes are separated by a distance of $\lambda/2$. Similarly the standing wave has a minimum amplitude points when $\sin kx = 0$ or when

$$kx = 0, \pi, 2\pi, 3\pi - - - - - n\pi \qquad 8.21\,(a)$$

substituting for $k = \frac{2\pi}{\lambda}$. in Eq. 8.21 (a) we get

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots, \frac{n\lambda}{2} \ldots\ldots\ldots \qquad 8.21(b)$$

Where n = 1,2,3..... These points of zero amplitude called nodes, are also separated by $\frac{\lambda}{2}$. The distance between a node and an antinode is $\frac{\lambda}{4}$.

8.12 FUNDAMENTAL FREQUENCY AND HARMONICS

We consider a stretched string of length L fixed at both ends as shown in Fig. 8.13 (a). Standing waves are set up by a continuous superposition of waves incident and reflected from the ends. The string has a number of natural patterns of vibrations called normal modes (as shown in Fig. 8.13 a,b,c, and d). Each of these modes has a characteristic frequency. The frequencies are easily calculated.

It is to be noted that the ends of the string must always be nodes because they are fixed. If the string is displaced at the middle point and released, the vibration is as shown in Fig. 8.13 (a). In this case, the middle point of the string has the maximum displacement, called anti-node. For this normal mode of vibration, the length of the string is equal to $\frac{\lambda_1}{2}$ which is distance between two neighbouring nodes. Since the length of the string is L, so we have

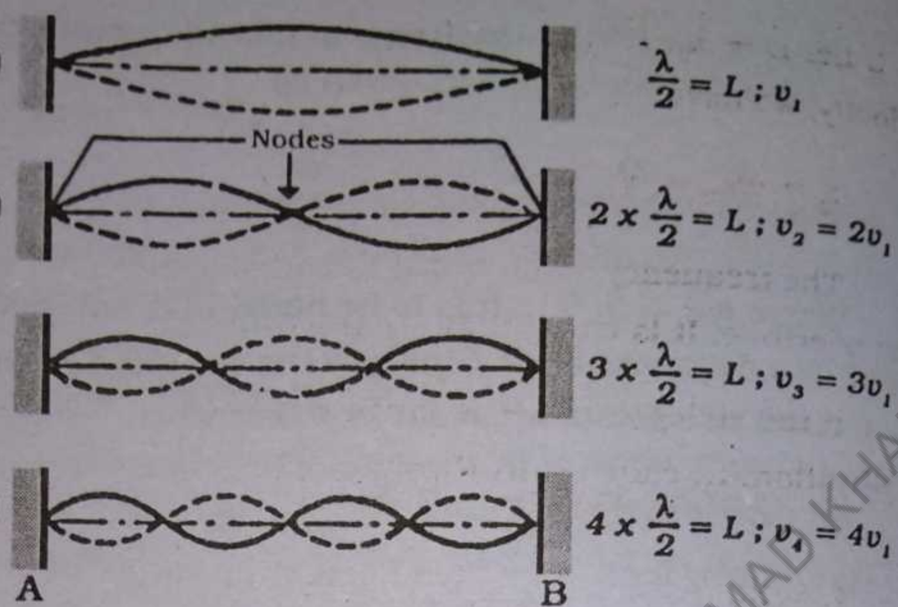


Fig.8.13 Standing waves on a stretched string. All such waves must have nodes at the termination points.

$$L = \lambda_1/2$$

or $\lambda_1 = 2L$

If f is the frequency of fundamental mode of vibration and ϑ is velocity of the wave then we have

$$f_1 = \frac{\vartheta}{\lambda_1} = \frac{\vartheta}{2L}$$

The frequency so obtained is called the fundamental frequency or the first hamonic of the standing wave.

The same string may be made to give out next higher order frequency instead of the fundamental, without altering the length of the string. This is done as follows:

A feather or a finger is very lighly pressed at a point half way between the middle point and the end of the string. The mode of vibration of the string is given in Fig. 8.13 (b). The string consists of two loops. The two ends of the string and its mid point are nodes whereas there are two anti-nodes each lying between two neighbouring nodes. Let λ_2 be the wavelength of this mode of vibration of the string and is equal

to L i.e. L = λ_2. If f_2 is the frequency of this mode and ϑ is the velocity, we have

$$f_2 = \vartheta/\lambda_2 = \vartheta/L = 2f_1$$

The frequency so obtained is called the second harmonic or first overtone. It is equal to twice the fundamental frequency.

If the string is made to vibrate in three loops, the frequency of vibration, f_3, called third harmonic or secong overtone, is given by

$$f_3 = 3f_1$$

In general if the string vibrates in n loops, the corresponding frequency f_n is given by

$$f_n = nf_1 \qquad 8.22$$

8.13 SONOMETER

A sonometer is a practical application of stretched strings for vibrating a portion of the string into desired number of loop . It consists essentially of a thin metalic wire stretched across two bridges A and B, on the top of a hollow, wooden sounding box about one metre long. One end of the wire is fastened to a peg at one end of the box. The other end passes over a smooth frictionless pulley fixed at the other end of the box. The pulley carries a scale pan, S, so that it can be loaded to have any desired tension in the string. In between the two bridges A and B, there is a movable bridge C sliding over a scale to adjust the length of the vibrating portion of the string. Here the bridges A,B,C always form nodes at their respective positions. If there is no bridge between the bridges A and B, then the simplest mode of vibration is one in which the portion of the string between A and B vibrates in one loop only.

The frequency of vibration of the string is minimum in this case. This corresponds to the fundamental frequency. This is denoted by f_1. Now place the bridge C just in the middle of A and B.

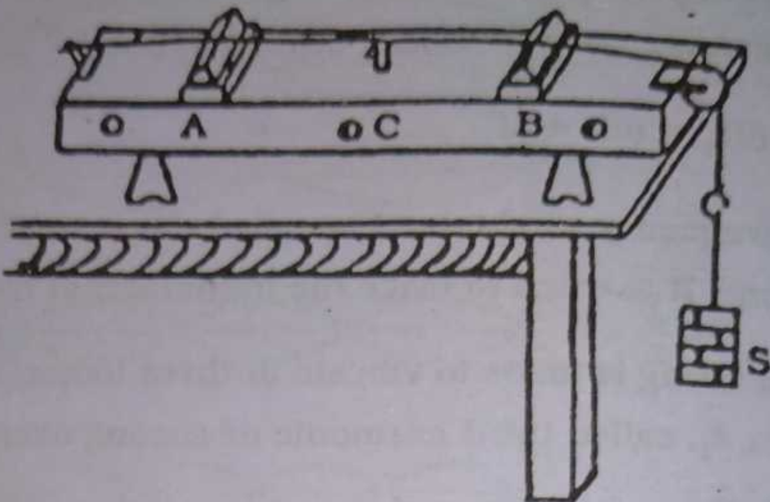


Fig. 8.14

The string will vibrate in two loops with a frequency $f_2 = 2f_1$. If the bridge is placed in such a position that the string vibrates in three loops, then the frequency excited in the string will be $3f_1$, and so on.

All the laws of transverse vibration of string can be verified using sonometer. If L is the length of viberating segment of the string. T is the tension and μ is the mass per unit length of the wire, then the frequency produced in the string is given by

$$f_1 = \frac{n\vartheta}{\lambda} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \quad \because \quad \vartheta = \sqrt{\frac{T}{\mu}} \qquad 8.23$$

Where n = 1, 2, 3, ... i.e. the frequencies are integral multiple of the fundamental frequency. The $\vartheta = \sqrt{T/\mu}$ represents tne speed of the wave on the string.

The above equation shows that

(i) the frequency produced in the string for a given tension is inversely proportional to its length i.e

$$f \propto \frac{1}{L}$$

(ii) for a string of given length and material, the frequency varies directly as a square root of the tension T i.e

$$f \propto \sqrt{T}$$

(iii) The frequency of vibration of strings of the same length and subjected to the same tension varies inversely as the square root of the mass per unit length, μ, of the string i.e.

$$f \propto \frac{1}{\sqrt{\mu}}$$

8.14 LONGITUDINAL WAVES

A second type of waves, called longitudinal waves are the waves in which the particles of the disturbed medium undergo displacement in a direction parallel to the direction of wave motion. These waves are produced in substances which are elastic and compressible like gases and wire springs. Sound waves are longitudinal which result from the disturbance of the medium. The disturbance corresponds to a series of high and low pressure regions that travel through air or through any other material medium with a certain velocity. A longitudinal pulse can be easily produced in a spring. Fig.8.15 represents a spring whose turns are large and one of whose ends is supported from a hook in the wall. If you compress a few turns of the spring near its left end, these move slightly and compress those just ahead and these in turn squeez together the turns still further and thus a pulse. or compression wave, goes along the spring.

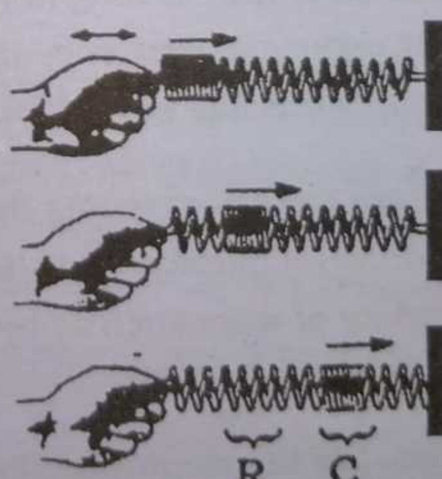


Fig. 8.15 A longitudinal pulse along a stretched spring. The disturbance of the medium (the displacement of the coils) is in the direction of the wave motion. For the starting motion described in the text, the compressed region C is followed by an extended region, R.

Next let the left end of the spring by given a quick pull so that the turns nearly are drawn apart for an instant. Then the adjacent turns will be pulled apart, one after another, untill this disturbance reaches the right end. Thus it is clear that any push or pull given to the spring at one end is transmitted as a push or pull to the other end. Waves of this sort in which the particles of the transmitted medium move back and forth in the direction of propagation of the waves are called compressional, longitudinal or pressure waves.

Sound waves are the most important example of longitudinal waves. They can travel through any material medium i.e. gases, solids and liquids with a speed which depends upon the properties of the medium. As sound waves travel through a medium, the particles of the medium vibrate along the direction of propagation of the wave motion. This is in contrast to a transverse wave motion in which the particles of the medium vibrate in a direction perpendicular to the direction of wave motion. The displacement that occurs as a result of sound waves involve the longitudinal displacement of the individual molecules from the equilibrium position. This results in a series of high and low pressure regions called compressions and rarefactions, respectively. Graphically the longitudinal wave is represented like transverse wave.

There are three categories of longitudinal mechanical waves which cover different regions of frequency (i) Audible waves (ii) Infrasonic waves and (iii) Ultrasonic waves.

Audible waves are the sound waves that lie within the range of sensitivity of human ear, typically, 20 H_z to 20,000 H_z. They can be generated in a variety of ways such as by musical instruments, human vocal chord and loud speakers.

Infrasonic waves, are the longitudinal waves with frequencies below audible range that is below 20 Hz. Earth quake waves are an example of infrasonic waves.

Ultrasonic waves are the longitudinal waves with frequencies above the audible range that is above 20 KHz. They can be generated by inducing vibrations in a quartz crystal with an applied alternating electric field.

8.15 SPEED OF SOUND WAVES

Newton's Formula for the speed of sound waves:

As we know, sound waves are compressional waves which propagate through a compressible medium such as air. The speed of such compressional waves depends upon the compressibility and the inertia of the medium. If the compressible medium has a bulk modulus denoted by B and density (inertial property) denoted by ρ, the speed of sound in that medium is given by

$$\vartheta = \sqrt{B/\rho} \qquad 8.24$$

The expression given by Eq. 8.24 is known as Newton's formula for the speed of sound waves.

Note that the bulk modulus, B, is the ratio of the change in pressure, Δp, to the resulting fractional change in volume, $-\Delta v/v$

$$B = -\Delta p / \left(\frac{\Delta v}{v}\right) \qquad 8.25$$

Here Δv is the change in original volume V. The ratio $(\Delta p/\Delta v)$ is always negative because Δv decreases as Δp increases and vice versa. This shows that B is always positive. In fact the speed of all mechanical waves can be expressed in a general form,

$$\vartheta = \sqrt{\text{elastic property/inertial property}}$$

Table 8.1
speed of sound in various media

Meduim	ϑ(m/s)
GASES	
Air (o°c)	331
Air (100°c)	336
Hydrogen (o°c)	1286
Oxygen (o°c)	317
Helium (o°c)	972
LIQUIDS at 25°c	
Water	1493
Methyl alcohol	1143
Sea water	1533
SOLIDS	
Aluminium	5100
Copper	3560
Iron	5130
Lead	1322
Vulcanized rubber	54

8.16 LAPLACE'S CORRECTION

The Newton's formula (Eq 8.24) was obtained on the assumption that the compressions and rare-factions take place at constant temperature. This kind of process is called an isothermal process and Boyle's Law is obeyed throughout the change from the initial pressure and volume to the final pressure and volume. Under this condition the bulk modulus B is equal to the pressure of gas. Therefore

$$\vartheta = \sqrt{P/\rho}$$

However, in a sound wave, the wave motion is so rapid and the heat conductivity is so low that there is insufficient time for the heat produced in the compressed regions to be conducted to the rarefied regions. This means that the process in no more isothermal. Therefore compressions and rare-factions occur adiabatically

(A process in which heat does not flow into or out of the system) and not isothermally as was assumed by Newton.

For such a case, the bulk modulus of the gas is γ times the pressure of the gas where γ is the ratio of molar specific heat of gas at constant pressure (Cp) to the molar specific heat at constant volume (Cv). The speed of sound is thus.

$$\vartheta = \sqrt{\gamma p/\rho} \tag{8.26}$$

This is called Laplace's correction.

If we use the ideal gas law PV = nRT then we can rewrite Eq 8.26 in the form

$$\vartheta = \sqrt{\frac{\gamma RT}{M}} \tag{8.27}$$

Where M is the molecular mass of the gas in units of kg/mole, n is number of moles R is universal gas constant and has value 8.314J/mol-K and T is temperature expressed on kelvin scale.

Using Eq 8.27 we can calculate the velocity of sound in air at 0°C. We know air consists of approximately 80% of Nitrogen and 20% Oxygen. The masses of Nitrogen and Oxygen molecules are 28 a.m.u and 32 a.m.u respectively. Therefore the mean molecular mass for air is:

Mean molecular mass for air = 0.8 x 28 a.m.u + 0.2 x 32 a.m.u = 28.8 a.m.u

Then M = 28.8 g/mole = 0.0288 kg/mole

The velocity of sound at 0°C is

$$\vartheta = \sqrt{1.40 \times (8.314\text{J/mole}-\text{K}) \times (273\text{K}) / 0.0288\text{ kg mole}}$$

$$\vartheta = 332\text{ m/s at }0^{\circ}\text{C}$$

At any other temperature T, the speed of sound in air can be obtained by multiplying this result by $\sqrt{T/273}$. For example, at an altitude of 10000 ft (3.05km), the temperature is about $-50^{\circ}C$ or 223K therefore

$$\vartheta = (332\ m/s) \times \sqrt{\frac{223}{273}} = 300\ m/s$$

8.17 MUSICAL SOUND AND NOISE

We perceive some sound as noise and some sound as music. For example the slamming of a door or the rumbling of a truck is considered as a noise, whereas, the sound produced by a piano or a guitar is regarded as musical. These two types of sound are represented in form of graphical patterns as shown in Fig.8.16 (a) and (b). The Fig.8.16 (a) represents an irregular, non symmetric, random fluctuations producing disagreeable sensation hence called noise: The Fig 8.16(b) shows a regular, symmetric and periodic fluctuation, which produce a smooth pleasant sensation hence called music. The physical difference between these two types of sound is better understood by studying their spectra. The spectrum of noise and the spectrum of musical sound are sketched in Fig. 18.16 (c) and Fig. 18.16 (d) respectively the spectrum of noise contains frequency components which are not harmonically related i.e. the frequency components are not integral multiple of the fundamental. Besides this the amplitude of each frequnecy components are mostly not symmetric.

The spectrum of musical sound contains frequency components which are harmonically related i.e. the frequency components are integral multiple of the fundamental. Besides this the fluctuations in amplitude are symmetric.

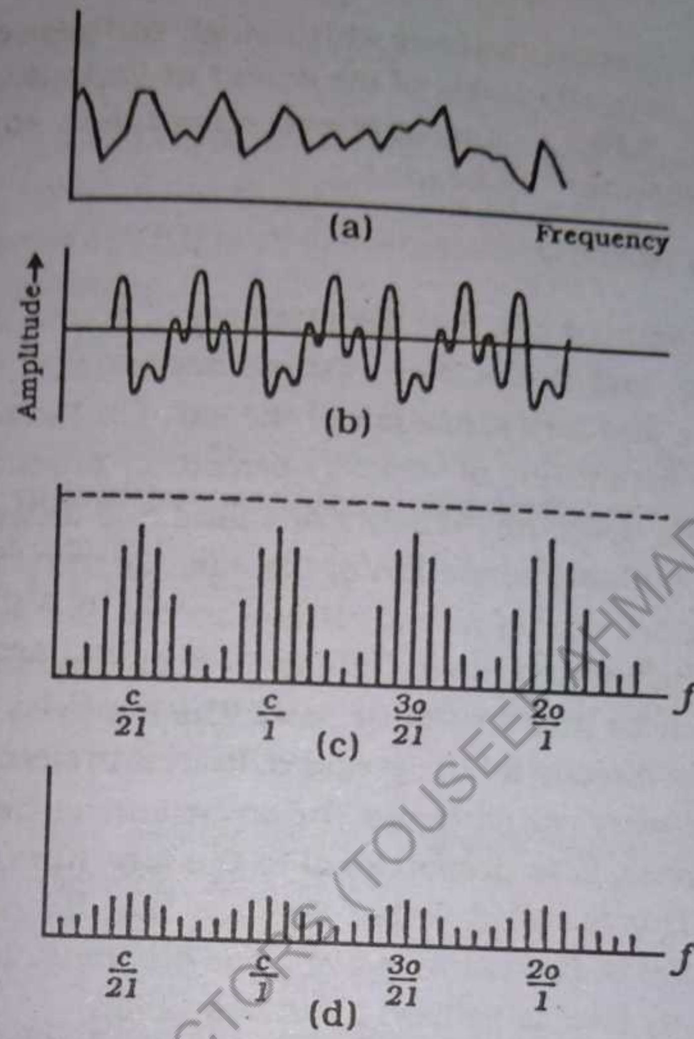


Fig. 8.16 (a) , (b) The wave pattern of a violin (c) Power spectrum of noise (d) power spectrum of musical signal.

Characteristics of a musical sound

Musical sounds or tones can be distinguished from one another by the following characteristics:

(i) Intensity (loudness)

(ii) Pitch or frequency

(iii) Quality

Intensity of sound is measured in terms of the energy of the sound wave, passing every second through a unit area perpendicular to the direction of propagation of sound and is measured in watt/m^2. For example, if one talks as loudly as possible, the average speech power is about 10^{-3} W/m^2; and if one speaks in as

weak a voice as possible (without whispering), the power drops to 10^{-7} W/m^2. The intensity levels of the sound at various frequencies are shown in Fig 8.16 . The figure also shows that threshold of pain and threshold of hearing.

(I) Loudness

It may be pointed out that the intensity of sound is purely a physical quantity and it can be measured accurately. It does not depend upon the auditory sensation of the ear. On the other hand Loudness is the magnitude of auditory sensation produced by the sound. It depends upon the intensity of sound and as well as it depends upon the auditory sensation of the ear. A normal human ear is very sensitive detector of sound. It can record the least intense sound (10^{-12} W/m^2) which is one billionth of the maximum sound intensity that can be heard without pain. The loudness of a sound does not increase directly as the power delivered to the ear increases but seems to vary roughly as the logarithm of the power. It means the loudness, L, is proportional to the logarithm of intensity I i.e. $L \propto \log I$. This is called Weber Fechner law. If I and I_o represent the intensities of two sound waves, the difference in loudness, known as intensity level is defined by the equation

$$\text{intensity level} = P = \log \frac{I}{I_o} \qquad 8.28(a)$$

where I_o is equal to 10^{-12} Wm^{-2} corresponding roughly to the fainstest sound that can be heard. This is regarded as arbitrary reference level. If I is taken ten times of I_o ($I = 10\, I_o$), then the intensity level is equal to one unit. This unit of intensity level is called a bel (B).

A bel is a rather large unit, so we usually refer to sound intensity levels in terms of the decibel (dB): $1\text{dB} = \frac{1}{10}$ B. thus

$$\beta = 10 \log \frac{I}{I_o} \qquad 8.28(b)$$

When $\frac{I}{I_o} = 1, \beta = 0$ dB

$\frac{I}{I_o} = 10, \beta = 10$ dB

$\frac{I}{I_o} = 10^2, \beta = 20$ dB

The loudness of a sound is roughly correlated with the intensity level of the note. For a long time it was accepted that the correlation was perfect. But there is now no doubt that decibel scale of intensity level no more correlates with the subjective sensation of loudness. A power law between loudness and intensity level of the note has been proposed.

$$L = K\left(\frac{I}{I_o}\right)^{0.3}$$

Where K is an arbitrary constant which can be evaluated after defining the unit of loudness 'sone'. The sone is defined as

1 sone = 40 dB at 1000 H_z

$\frac{I}{I_o} = 10^4$ when $\beta = 40$ dB, therefore K = 1/16.

$$\therefore L = \frac{1}{16}\left(\frac{I}{I_o}\right)^{0.3} \qquad 8.28(c)$$

The sone scale of loudness has been recognised internationally, and being practised.

(ii) Pitch

It is defined as the sensation that sound produces in the ear of a listener and is clearly related to the frequency of sound. Frequency and pitch are both measured in Hertz (Hz). Thus greater the frequency the greater the pitch and lower the frequency lower the pitch. The pitch of sound produced by various physical instruments usually depends upon the natural resonant frequency of the instrument. This is our common experience that sound of a sparrow is shrill because of its high pitch since it has a high frequency. Whereas the sound of a lamb is grave due to its low pitch depending upon its low frequency.

(iii) Quality

Quality or timbre is a characteristic of a musical sound. It is quality which enables us to distinguish between notes of the same pitch and intensity when played on different instruments or sung by different voices. Even instruments of the same kind may yield

notes of different quality. For example, it is the quality of the tones produced by two violins which makes great difference in value between the instruments. We recognise the voice of a friend over the telephone by quality.

Helmholtz was the first to discover the cause of difference in musical tones which is called quality.

The difference in the sound produced by two notes of the same pitch and intensity is due to the difference in their resultant waveforms. The resultant wave form of any sound is obtained by

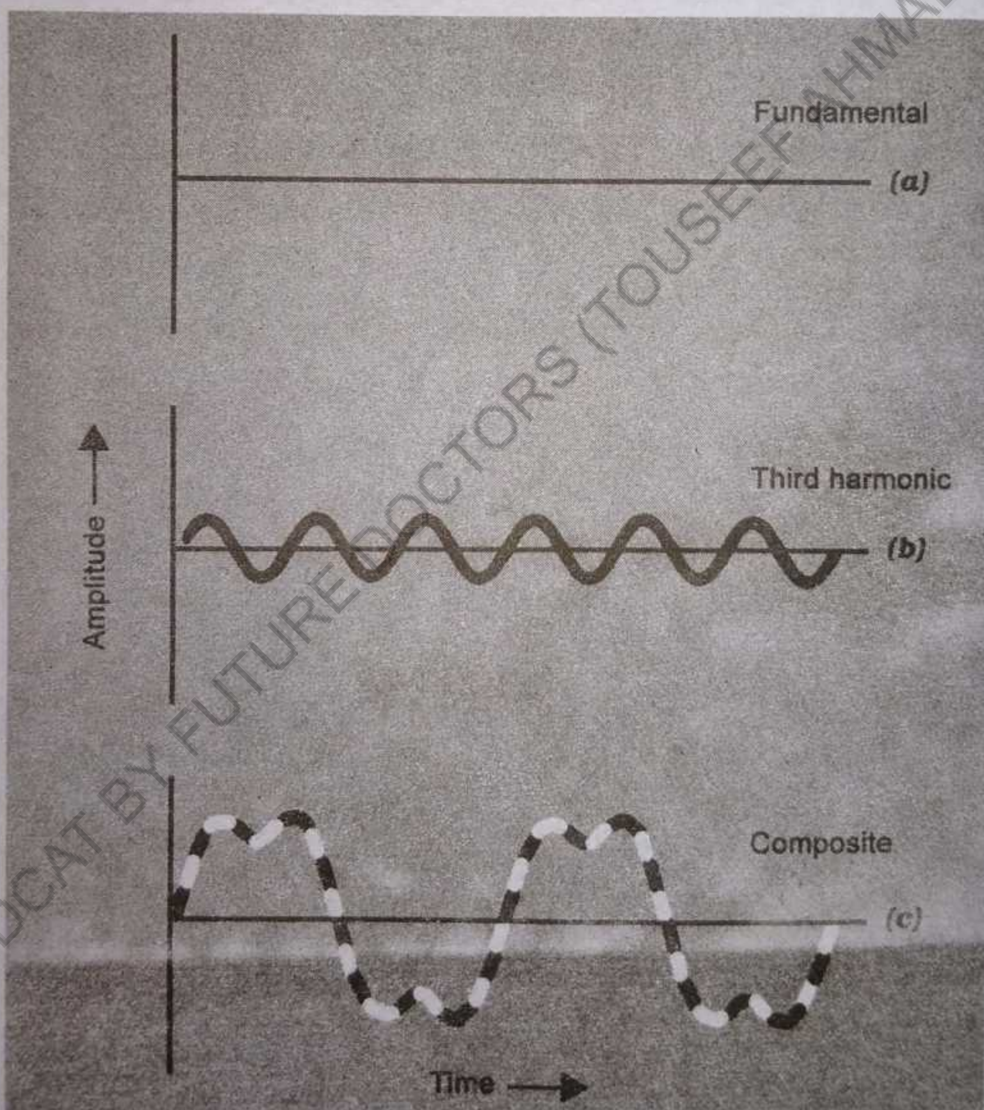


Fig. 8.17 The combination or superposition of two waves. (a) the fundamental and (b) the third harmonic.

combining the amplitudes of fundamental and the harmonics of the given sound.

The resultant waveforms are formed due to combination of fundamental and harmonics as shown in Fig.8.17 (a) and 8.17(b). In Fig: 8.17 (c), the fundamental and third harmonic are combined together to give a resultant wave form.

8.18 SUPERPOSITION OF SOUND WAVES

According to principle of superposition when two or more waves in the same (linear) medium travel the net displacement of the medium caused by the resultant wave at any point is equal to the algebraic sum of the displacements of all the waves. We shall apply this principle to two harmonic waves travelling in the same direction in a medium. The two waves are travelling to the right and have the same frequency, same wavelength and same amplitude, but they differ in phase, we can express their individual wave function displacements as

$$y_1 = A_o \sin(kx - \omega t) \text{ and } y_2 = A_o \sin(kx - \omega t - \phi)$$

Hence the resultant wave. function displacement is given by

$$y = y_1 + y_2 = A_o [\sin(kx - \omega t) + \sin(kx - \omega t - \phi)]$$

We make use of following trigonometric identity to simplify it

$$\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right)$$

Let $\alpha = kx - \omega t$ and $\beta = kx - \omega t - \phi$. we get

$$y = \left(2A_o \cos\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right) \qquad 8.29$$

There are several important features of this result. It can be easily seen that the resultant wave function y is also harmonic and has the same frequency and wavelength as the individual waves. The amplitude of the resultant wave is $2A_o \cos\frac{\phi}{2}$ and its phase is equal to $\frac{\phi}{2}$. If the phase constant ϕ is zero, then $\cos\frac{\phi}{2}$ is unity

and the amplitude of the resultant wave is $2A_o$. This means that the amplitude of the resultant wave is twice as large as that of either of individual wave having the same wave length. In this case, the waves are said to interfere constructively that is the crests of one fall on the crests of the other and similar is the case for troughs as shown by the dotted lines in Fig.8.18 (a).

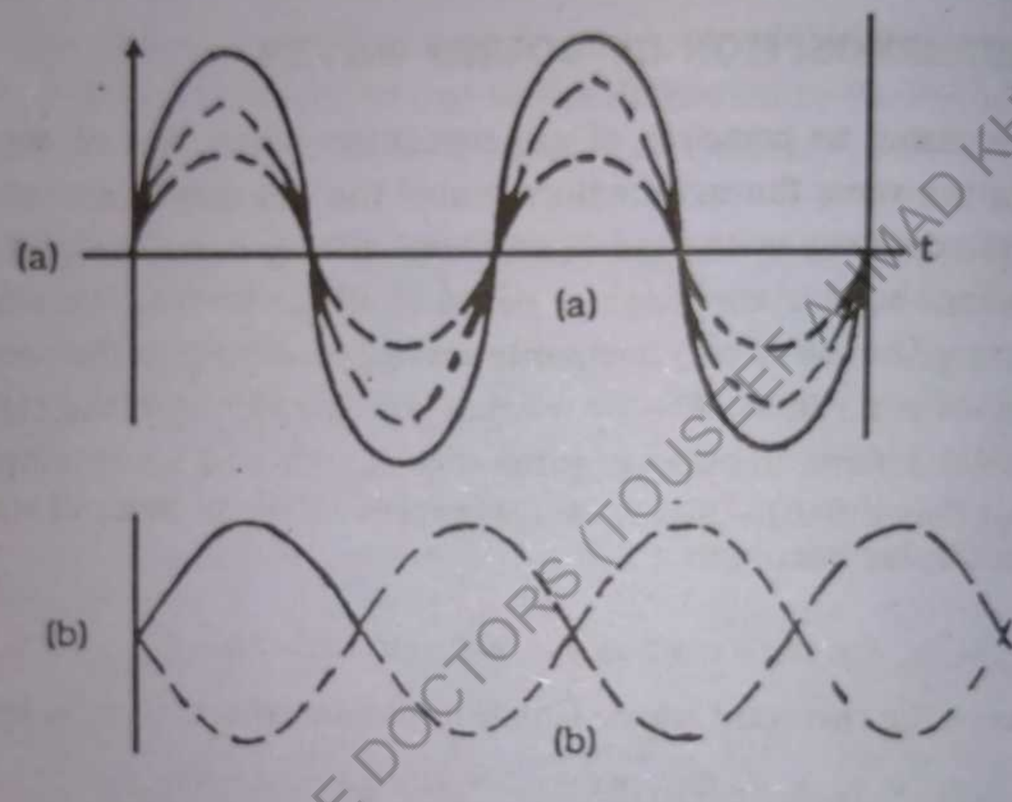


Fig. 8.18 (a), (b)

In general, constructive interference takes place where $\cos\frac{\phi}{2} = \pm 1$, or where $\phi = 0, 2\pi, 4\pi, \ldots\ldots, 2n\pi$ Where n is an integer, on the other hand if $\phi = \pi$ radians (or any odd multiple of π) then $\cos\frac{\phi}{2} = \cos\frac{\pi}{2} = 0$ and the resultant wave has zero amplitude every where. In this case, the two waves are said to interfere destructively i.e. the crests of one wave coincide with the troughs of the second and vice versa as shown in Fig.8.18 (b) and their displacement cancel at every point.

8.19 BEATS

So far we have been discussing the interference or superposition of two waves having the same frequency and travelling in op-

posite direction. We now consider another type of interference effect which results from the superposition of two waves having slightly different frequencies and travelling in the same direction. Under this condition when two waves are observed at a given point, they are periodically in and out of phase, i.e. there is an alternation in time between constructive and destructive interference. Thus, we refer to this phenomenon as interference in time. For example if two tuning forks of slightly different frequencies are struck on rubber pad, one hears a sound of changing intensity, called beats. Hence we can define beats as the periodic variation in intensity at a given point due to superposition of two wave having slightly different frequencies. The number of beats that one hears per second(or beats) is equal to the difference in frequency between two sources. The maximum beat frequency that the human ear can detect is about 7 beats per second. When the beat frequency (number of beats produced per second) is greater than seven, we can not hear them clearly. One can use this effect to tune instrument such as piano, by beating a note against a reference tone of known frequency. The string can then be adjusted to equal the frequency of the reference by tightening or lossening it until no beats are heard.

Now consider two waves with equal amplitude travelling through a medium in the same direction having slightly different frequencies f_1 and f_2. We can represent the displacement that each wave would produce at a point as:

$$y_1 = A_o \cos 2\pi f_1 t \text{ and } y_2 = A_o \cos 2\pi f_2 t$$

$$y = y_1 + y_2 = A_o (\cos 2\pi f_1 t + \cos 2\pi f_2 t)$$

$$= 2A_o \cos 2\pi t \frac{(f_1 - f_2)}{2} \cos 2\pi t \frac{(f_1 + f_2)}{2} \qquad 8.30$$

Here y_1 and y_2 are the instantaneous displacements caused by the individual wave as shown in Fig.8.19 (a) and y is the resultant displacement caused by the two waves together as shown in Fig. 8.19 (b)

The resultant displacement has an effective frequency equal to the average frequency, $(f_1 + f_2)/2$, and an amplitude given by

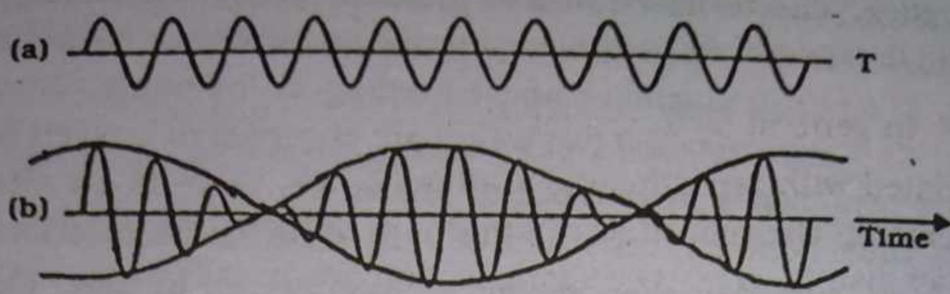


Fig. 8.19 Two waves with slightly different frequencies combine to produce beats.

$$A = 2A_o \cos 2\pi \left(\frac{f_1 - f_2}{2}\right) t \qquad 8.31$$

That is, the amplitude varies in time with frequency given by $(f_1 - f_2)/2$. When f_1 is closed to f_2, the amplitude variation is shown as illustrated by the envelope (broken line) of the resultant wave form. Fig.8.19 (b)

Note that a beat, for a maximum in amplitude, will be detected whenever

$$\cos 2\pi \left(\frac{f_1 - f_2}{2}\right) t = \pm 1$$

That is, there will be two maxima in each cycle. Since the amplitude varies with the frequency as $(f_1 - f_2)/2$, the number of beats per second, or the beat frequency f_b, is twice this value. That is

$$f_b = f_1 - f_2 \qquad 8.32$$

For example if two tuning forks vibrate individually at frequencies of 439Hz and 443 Hz, the resultant sound wave of combination would have a frequency of 441Hz and a beat frequency of 4Hz. (The listener would hear the 441 Hz sound wave go through an intensity maximum four times every second.)

8.20 ACOUSTICS

The study of production and properties of sound is called

acoustics. The term is also used to describe the way in which sound is reproduced in practical situations.

In general by acoustics we mean its second statement which is related with reproduced sound so we limit our discussion to this statement. The reproduced sound may be indistinct and confusing. It may also be clear and distinct. The quality of reproduced sound depends on several factors which we will enlist later on.

To give a clear concept of acoustics let us suppose that sound waves are produced in a big hall. These sound waves spread out, scatter and strike the surfaces of the walls, the floor and the ceiling of the hall and also the surfaces of various objects present in the hall. Some of the sound energy is absorbed by these surfaces and some is reflected. The reflected parts (consisting of many reflected sound waves) travel back and recombine to form undesirable echoes. These echoes interfere with the original sound waves coming directly from the source. The result is that we have an indistinct and unintelligible sound and hence give rise to bad acoustics.

Conditions for good acoustics

(1) The loudness of each separate syllable should be sufficiently large.

(2) The decay period of each syllable should be small so that the succeeding syllable can be heard clearly.

(3) Echoes should be just sufficient to maintain the continuity of sound.

(4) The hall should have some open windows. Sound absorbing soft porous materials like cloth, cork, asbestos etc or heavy curtains should be placed in the hall at various places so as to avoid much reflection.

(5) For good acoustics of a hall reverberation should not be too small otherwise that will be away instantaneously and will give a dead effect to the hall.

8.21 DOPPLER EFFECT

When a source of sound or a listener, or both are in motion relative to the medium (air), the frequency and hence the pitch of the sound, as heard by the listener, is in general not the same as when listener and source are at rest. This phenomenon is referred to as the **Doppler Effect**. A common example is a train while whistling passes near you. A considerable change in the pitch of the sound is heard. When the train is approaching, the pitch of the sound increases whereas the pitch of the sound decreases when the train is moving away. A similar change in the pitch also occurs when moving listener passes a stationary automobile horn or siren.

Obviously, there are three general possibilities to discuss the Doppler effect, namely:

I. When the listener is moving and the source is at rest.

II. When the source is moving and the listener is at rest.

III. When both the source and the listener are moving.

I(a). Suppose the listener is moving toward a stationary source as shown in Fig.8.20 with speed ϑ_o. Suppose further that the source emits a wave with frequency υ and wavelength $\lambda = \vartheta/\upsilon$. The Fig.8.20 shows several wave crests separated by equal distances 1λ (1 wavelength). The waves approaching the moving listener have a speed of propagation relative to the listener $\vartheta + \vartheta_o$ Thus the frequency υ' heard by the listener is

$$\upsilon' = \frac{\vartheta + \vartheta_o}{\lambda} = \frac{\vartheta + \vartheta_o}{\vartheta/\upsilon}$$

$$= \left(\frac{\vartheta + \vartheta_o}{\vartheta}\right)\upsilon = \left(1 + \frac{\vartheta_o}{\vartheta}\right)\upsilon$$

$$\upsilon' = \upsilon + \left(\frac{\vartheta_o}{\vartheta}\right)\upsilon \qquad 8.33$$

Therefore the listener moving toward a source at rest, detects a larger frequency and hence higher pitch. Consequently, the change in pitch in this case is

$$\nu' - \nu = \left(\frac{\vartheta_o}{\vartheta}\right)\nu \qquad 8.34$$

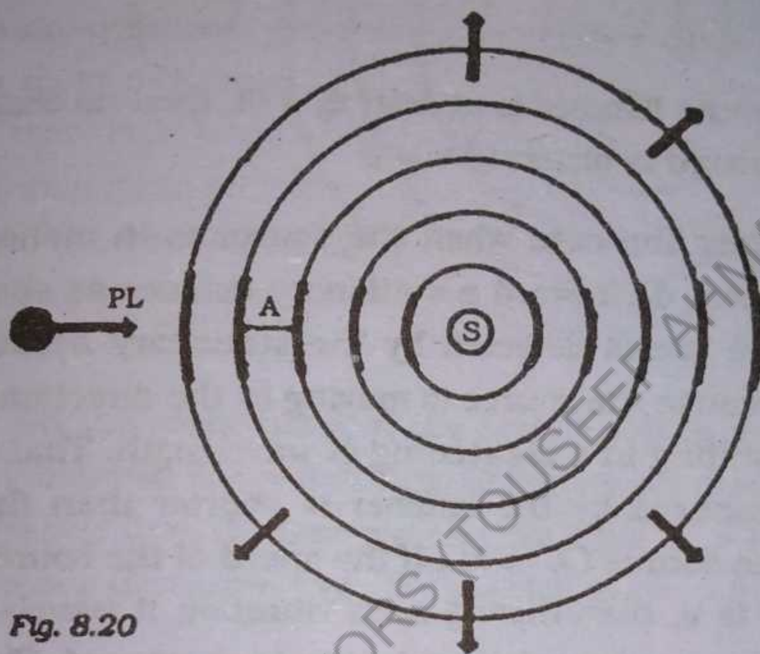


Fig. 8.20

I (b). Similarly, a listener moving away from the stationary source (ϑ_o<0) hears a lower pitch and the frequency detected by the listener in this case is

$$\nu' = \left(1 - \frac{\vartheta_o}{\vartheta}\right)\nu$$

$$\nu' = \nu - \left(\frac{\vartheta_o}{\vartheta}\right)\nu \qquad 8.35$$

Consequently, the change in pitch in this case is

$$\nu' - \nu = -\left(\frac{\vartheta_o}{\vartheta}\right)\nu$$

Hence, the general relation holding when the source is at

rest with respect to the medium and observer is moving through it, is given by

$$\upsilon' = \upsilon \pm \left(\frac{\vartheta_o}{\vartheta}\right)\upsilon = \left(\frac{\vartheta \pm \vartheta_o}{\vartheta}\right)\upsilon \qquad 8.36$$

Where the positive sign refers to the motion toward the source and negative sign refers to the motion away from the source.

Notice when the listener is at rest($\vartheta_o = 0$), then no change in the frequency of sound is observed $\upsilon' = \upsilon$

II (a). Now consider the case when the source is in motion and moving with a speed, ϑ_s, toward a stationary listener as shown in Fig.8.21. The wave crests detected by the stationary listener are closer together because the source is moving in the direction of the outgoing wave resulting in a shortening of wavelength. That is, the wavelength λ' measured by the listener is shorter than the true wavelength λ of the source ($\lambda' < \lambda$) . If the speed of the source is ϑ_s and its frequency is υ, then during each vibration it travels a distance $\frac{\vartheta_s}{\upsilon}$ and consequently each wavelength is shortened. Thus the wavelength of the sound arriving at the listener at rest is

$$\lambda' = \frac{\vartheta}{\upsilon} - \frac{\vartheta_s}{\upsilon} = \frac{1}{\upsilon}(\vartheta - \vartheta_s)$$

therefore, the frequency of the sound heard by the listner (at rest is increased and is given by

$$\upsilon' = \frac{\vartheta}{\lambda'} = \frac{(\vartheta\upsilon)}{(\vartheta - \vartheta_s)}$$

$$\upsilon' = \frac{\upsilon}{1 - \left(\frac{\vartheta_s}{\vartheta}\right)} \qquad 8.37$$

The Eq. 8.37 indicates an increase in the frequency of the sound heard by stationary listener.

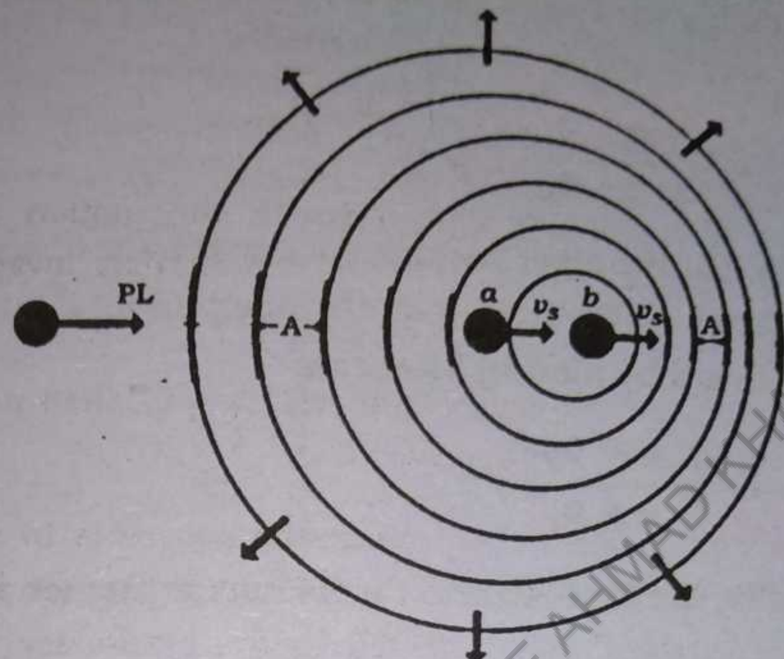


Fig. 8.21

II(b). On the other hand, if the source moves away from the stationary listener, the wavelength of the sound arriving at the listener is greater than the true wavelength λ (i.e $\lambda' > \lambda$)) and the listener detects a decreased frequency which is given by

$$\upsilon' = \frac{\upsilon\,\vartheta}{(\vartheta + \vartheta_s)} = \frac{\upsilon}{\left(1 + \frac{\vartheta_s}{\vartheta}\right)} \qquad 8.38$$

The Eq.8.38 indicates a decrease in the frequency of the sound heard by the stationary listener. Thus in general relation valid for the listener at rest and for the source in motion with respect to the medium is given as

$$\upsilon' = \left(\frac{\vartheta}{\vartheta \pm \vartheta_s}\right)\upsilon = \frac{\upsilon}{\left(1 \pm \frac{\vartheta_s}{\vartheta}\right)} \qquad 8.39$$

where the minus sign refers to the motion of the source toward the stationary observer and plus sign indicates the motion of the source away from the stationary observer.

Notice that when the source is at rest ($\upsilon_s = 0$). then no change the frequency of sound is observed (i.e $\upsilon' = \upsilon$)

III(a). If the source and the observer are approaching along the line joining the two in the direction toward the each other, then the frequency heard by moving listener is

$$\upsilon' = \left(\frac{\vartheta + \vartheta_o}{\vartheta - \vartheta_s}\right)\upsilon \qquad 8.40$$

III(b). On the other hand, if source and the observer are moving away from each other along the line joining the two, then the frequency heard by moving listener is

$$\upsilon' = \left(\frac{\vartheta - \vartheta_o}{\vartheta + \vartheta_s}\right)\upsilon \qquad 8.41$$

Finally, the Eq.8.40 and Eq.8.41 are expressed together as

$$\upsilon' = \left(\frac{\vartheta \pm \vartheta_o}{\vartheta \mp \vartheta_s}\right)\upsilon \qquad 8.42$$

Notice that Eq.8.42 reduces to Eq.8.36 when $\vartheta_s = 0$ (i.e. when the source is at rest) and when $\vartheta_o = 0$ reduces to Eq.8.39. Thus the validity of the Eq.8.42 is checked.

Notice when both the listener and source are at rest ($\vartheta_o = 0$, $\vartheta_s = 0$), then no change in the frequency of sound is observed ($\upsilon' = \upsilon$).

Fig.8.20 shows what happens when the speed of source (ϑ_s) through the medium is less than the speed of the sound wave (ϑ) in that medium i.e. $\vartheta_s < \vartheta$. Now we would like to examine what happens when the speed of the source ϑ_s is comparable in magnitude to ϑ ($\vartheta_s \equiv \vartheta$)? In this case the source keeps pace with the outgoing spherical waves in the direction of medium since $\vartheta_s \equiv \vartheta$. These waves can never move ahead of the source and they continue to pile up. As the process continues (i.e. wave are formed) the pile region extends farther and farther from the source in the direction perpendicular to its motion. This happens because the linear relationship between the restoring force and displacement (assumed

uptil now) does not remain true. The speed of the wave propagation (characteristics of the medium) is no longer the normal phase velocity. Components of the motion perpendicular to the line joining source and observer also contributes to the Doppler effect at these high speeds. Consequently, a wavefront that is a sheet or plane is formed as shown in Fig. 8.22.

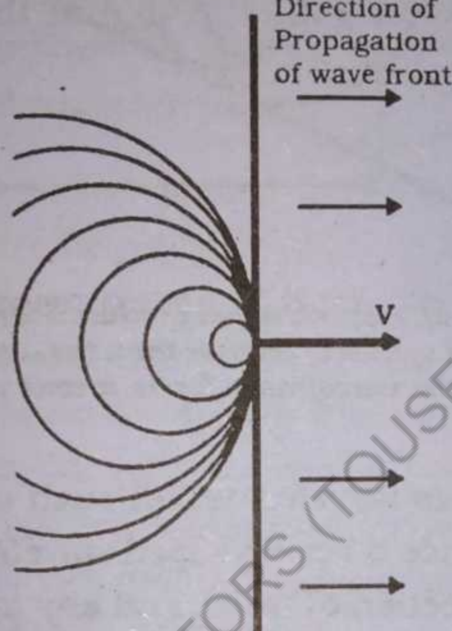


Fig. 8.22 When the source moves with a speed exactly equal to the wave speed, the waves pile up and form a plance that extends perpendicular to the direction of motion of the source.

Next we would like to see what happens when ϑ_s exceeds the wave speed i.e. $\vartheta_s > \vartheta$ Obviously, in this case, the source runs ahead of the outgoing waves and in such a case the pile up of the waves produces wavefront which takes the shape of a cone with the moving object at its apex as shown in Fig 8.23.

The situation is best explained by considering the source at S_o (Fig.8.23) at time t = 0, and some later time t, the spherical wavefront centered at S_o travels a distance ϑt (the radius of the wavefront). Whereas the source reaches at S_n after travelling a distance $\vartheta_s t$. At that instant the source is at S_n and the waves just beginning to be generated at this point have, wavefronts of zero radius. The line joining S_n and the wavefrom centerd on S_o is tangent

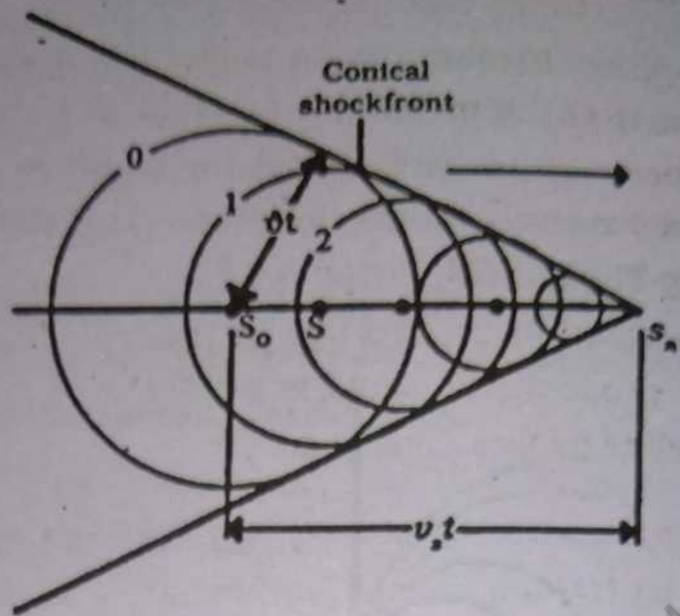


Fig.8.23 Representation of a shock wave produced when a source moves from So to Sn with a speed v_s that is greater than the wave speed, v, in that medium. The envelope of the wavefronts forms a cone whose half-angle is given by $\sin\theta = \vartheta/\vartheta_s$.

to all other intermediates wavefronts. All such tangent lines lie on a surface of a cone, hence a cone-shaped wavefront is formed. The angle θ between the direction of travel and any tangent line is given by

$$\sin\theta = \frac{\vartheta t}{\vartheta_s t} = \frac{\vartheta}{\vartheta_s}$$

In aerodynamics the ratio ϑ/ϑ_s is referred to as the **Mach number.**

Along the cone-shaped wavefront that is produced by a supersonic object, the air is highly compressed. This moving sheet of high pressure air is called a shock wave. Some examples are the bow wave from a speed boat on the water when the boat's speed exceeds the speed of water waves. Jet aircraft (F-16) travelling at a supersonic speed ($\vartheta_s > \vartheta$) produces a shock wave which results in loud explosion or sonic boom. These shock waves produced by supersonic jet can damage building when flying at low altitudes. In supersonic jets for example F-16, Mirage Fighter, Concorde airliner, a double boom is produced because two shock waves are formed (within time interval of 0.02s) one from the front and one from the rear of the aircraft.

APPLICATIONS OF THE DOPPLER EFFECT

(i) The Doppler effect provides a method for tracking a satellite. Suppose the satellite is emitting a radio signal (i.e an electromagnetic wave) of constant frequency fs. The frequency f_L of the signal received on the earth decreases as the satellite is passing. The received signal is combined with a constant signal generated in the receiver to produce beat. The beat frequency produces an audible note whose pitch changes as the satellite passes overhead.

(ii) The Doppler effect is used in measuring the speed of automobile by traffic police. A "radar gun" is fixed on police car. An electromagnetic signal is emitted by the radar gun in the direction of the automobile whose speed is to be checked. The wave is reflected from the moving automobile and received back. The reflected wave is then mixed with the locally generated original signal and beats are produced. The frequency shift is measured using beats and hence the speed of the automobile is determined.

(iii) Radars (Radio detection and ranging) are commonly used for civil and military purposes at civilian airports and military air-bases respectively. to detect the presence of any aircraft (foe or friend) in the airspace by evaluating the range. The Doppler Radar based on the principle of the Doppler Effect, is extensively used in the detection of aircraft speed and direction.

(iv) VOR (Very high frequency omni Range) is a guiding system usually installed at the airports to guide the incoming aircrafts toward the location of the airport. Nowadays big and modern airports for example Quaid-e-Azam International Airport, Islamabad Airport, are equipped with Doppler VOR whose principle is once again based on the Doppler effect and provides air effective and better guiding system to the

aircraft. The electromagnetic signal used here has frequency in the VHF range ($30MH_Z - 300MH_Z$).

(v) X-rays have been a major diagnostic tool of medicine, but recent years have seen the emergence of an alternate tool which is inherently safer than x-rays: ultra sound. Unlike x-rays, ultra sound radiations have not been reported to damage living cells so far. Measurements of internal reflections of ultra sound have facilitated the diagnosis of breast cancer and taking the heartbeats of fetuses and newborns.

An ultrasonic instrument called Homosonde uses the doppler effect of ultrasonic waves reflected from moving masses in the patient. The device is very sensitive for detecting blood flow and measures faint heartbeats in a very noisy environment where the use of stethoscope may not be reliable.

(vi) In similar manner, we can detect the motion of an objects under-water (for example a submarine) by employing sonar (ultrasonic waves) based on effective use of Doppler effect.

(vii) The Doppler effect for light is important in astronomy. Spectral analysis of light emitted by the elements in distant stars shows shift in the wavelength compared to light from the same element on earth. These shifts can be interpreted as Doppler shifts due to the motion of the stars. The shift is nearly always toward the longer wavelength, or red end of the spectrum, and is therefore called the red shift. Such observations have provided practically all the evidence for "expanding universe" cosmological theories, which represents the universe as having evolved from a great explosion several billion years ago in a relatively small region of space.

Example 8.1

A block with a mass of 0.1 kg is attached to a spring and placed on a horizontal frictionless table. The spring is stretched 20 cm when a force of 5N is applied. (i) Calculate spring constant (ii) Find the characteristic frequency and period of oscillation when the mass is set in motion.

(i) The applied force and displacement are related by

$$F = kx, \text{ therefore}$$

$$K = \frac{F}{x} = \frac{5N}{0.2m} = 25N\,m^{-1}$$

(ii) The characteristic frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{25N\,m^{-1}}{0.1kg}} = 2.52\text{ Hz}$$

The period of osillation is $T = \frac{1}{f}$

$$T = \frac{1}{2.52\text{ Hz}} = 0.397s$$

Example 8.2

A body of mass 2kg attached to a spring is displaced through 0.04 m from its equilibrium position and then released. If the spring constant is 200 Nm^{-1}, (i) find the period and frequency of vibration:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2kg}{200N\,m^{-1}}} = 0.628s$$

$$f = \frac{1}{T} = \frac{1}{0.628\text{ s}} = 1.59\text{ Hz}$$

$$\omega = 2\pi f = \sqrt{\frac{K}{m}} = \sqrt{\frac{200N\,m^{-1}}{2kg}} = 10s^{-1}$$

(ii) Calculate the maximum velocity attained by the vibrating body:

The maximum velocity occurs at the equilibrium position i.e. when x=0, for any x.

$$\vartheta_x = \pm\omega\,(x_o^2 - x^2)^{1/2}$$

So when $x = 0$,

$$\vartheta_x = \vartheta_{max} = \pm\,\omega x_o$$

$$= \pm(10s^{-1})\,(0.04) = \pm\,0.4\text{ m s}^{-1}$$

(iii) Compute the maximum acceleration

We know $a_x = -\omega^2 x$

the maximum acceleration occurs at the end of the path, i.e. when $x = \pm x_o$, therefore

$$a_{max} = \pm\,\omega^2 x_o = \pm\,(10s^{-1})^2\,(0.04) = \pm\,4.0\text{ m.s}^{-2}$$

Example 8.3

A body of mass 0.025 kg attached to a spring is displaced through 0.1m to the right of equilibrium position. If the spring constant is 0.4 Nm^{-1} and its velocity at the end of this displacement be 0.4 ms^{-1}, calculate (i) The total energy (ii) the amplitude of its motion (i.e. maximum displacement)

(i) Total energy:

The total energy at any instant during the motion is equal to the sum of potential and kinetic energy at that instant

$$E = \frac{1}{2}\,kx^2 + \frac{1}{2}\,m\vartheta^2$$

$$= \frac{1}{2}\,0.4(Nm^{-1})\,(0.1m)^2 + \frac{1}{2}\,(0.025\text{ kg})\,(0.4ms^{-1})^2$$

$$= 4\times10^{-3}\text{ J}$$

(ii) Amplitude:

Total energy is equal to the maximum potential energy. The maximum potential energy occurs at the end of motion where the kinetic energy is zero (the body is momentarily at rest).

$$E = K.E + P.E$$

$$E = 0 + (P.E)_{max} = \frac{1}{2} kx_o^2$$

$$\frac{1}{2} kx_o^2 = 4 \times 10^{-3} J$$

$$x_o^2 = \sqrt{\frac{2 \times 4 \times 10^{-3} J}{0.4 \; Nm^{-1}}} = 0.1414m$$

$$x_o = 0.1414m$$

Example 8.4

A simple pendulum completes one oscillation in 2 s. Calculate its length when $g = 9.8ms^{-2}$, the time period of simple pendulum is given by

$$T = 2\pi \sqrt{\frac{l}{g}}$$

$$l = \frac{T^2}{4\pi^2} \times g = \frac{4 \times 9.8 \; ms^{-2}}{4 \times (3.141)^2} = 0.992m$$

$$l = 0.992m$$

Example 8.5

A string 2 m long and of mass 0.004 kg is stretched horizontally by passing one end over a pulley and attaching a 1 kg mass to it. Find the speed of the transverse waves on the string and the frequency of the second and the fourth harmonics to which the string will resonate.

The speed of a transverse wave is given by equation

$$\vartheta = \sqrt{\frac{T}{m/l}} = \sqrt{\frac{Mg}{m/l}}$$

where $T = Mg$ (Tension in the string)

M = the Total mass suspended from the string = 1 kg

g = the acceleration due to gravity = 9.8 m s^{-2}

Substituting the various values, we get

$$\vartheta = \sqrt{\frac{1 \times 9.8}{0.004/2}}$$

$$\vartheta = 70 ms^{-1}$$

For resonance frequencies we know that

$$\upsilon_n = n\frac{\vartheta}{2l}$$

For second harmonic (υ_2), $n = 2$

$$\therefore \upsilon_2 = \frac{2 \times 70}{2 \times 2} = 35 \text{ hertz}$$

For fourth harmonic (υ_4), $n = 4$

$$\therefore \upsilon_4 = \frac{4 \times 70}{2 \times 2} = 70 \text{ hertz}$$

Example 8.6

A certain string resonates to several frequencies, the lowest of which is 50 hertz. What are the next four higher frequencies to which it resonates?

We have $\upsilon_n = n\frac{\vartheta}{2l}$ where $n = 1, 2, 3, \ldots\ldots$

Therefore all the higher frequencies are integral multiples of the lowest frequency which corresponds to $n=1$.

ing = 1 kg

$$\upsilon_1 = 50 \text{ hertz}$$

$$\upsilon_2 = 2 \times \upsilon_1 = 2 \times 50 = 100 \text{ hertz}$$

$$\upsilon_3 = 3 \times \upsilon_1 = 3 \times 50 = 150 \text{ "}$$

$$\upsilon_4 = 4 \times \upsilon_1 = 4 \times 50 = 200 \text{ "}$$

$$\upsilon_5 = 5 \times \upsilon_1 = 5 \times 50 = 250 \text{ "}$$

Example 8.7

The speed of a wave on a particular string is 24 m/s.If the string is 6 m long, to what frequencies will it resonate? Draw a picture of the string for each resonant frequency.

As $\lambda_n = 2l/n$ where n = 1, 2, 3, 4, 5, ...

we have $\lambda_1 = \dfrac{2 \times 6}{1} = 12\text{m}$

$$\lambda_2 = \frac{2 \times 6}{2} = 6\text{m}$$

$$\lambda_3 = \frac{2 \times 6}{3} = 4\text{m}$$

$$\lambda_4 = \frac{2 \times 6}{4} = 3\text{m}$$

As the frequency υ_n is given by

$$\upsilon_n = \frac{\vartheta}{\lambda_n}$$

$$\therefore \upsilon_1 = \frac{\vartheta}{\lambda_1} = \frac{24}{12} = 2 \text{ hertz}$$

$$\upsilon_2 = \frac{\vartheta}{\lambda_2} = \frac{24}{6} = 4 \text{ hertz}$$

lowest of to which

iples of

$$\upsilon_3 = \frac{\vartheta}{\lambda_3} = \frac{24}{4} = 6 \text{ hertz}$$

$$\upsilon_4 = \frac{\vartheta}{\lambda_4} = \frac{24}{3} = 8 \text{ hertz}$$

Example 8.8

A spring 4 m long resonates in four segments (node at both ends). The frequency of driving system on the spring is 20 hertz. Calculate the speed of the wave in the spring.

As the spring is vibrating in four segments, $l = 2\lambda$

$$\lambda = l/2 = \frac{4}{2} = 2\text{m}$$

Again $\vartheta = \upsilon \times \lambda$

$$= 20 \times 2$$

$$\vartheta = 40 \text{ m/s}$$

We can obtain the same result from the equation

$$\vartheta = n\frac{\vartheta}{2l}$$

$$\vartheta = \frac{2\upsilon \times l}{n}$$

$$\vartheta = \frac{2 \times 4 \times 20}{4} = 40 \text{ m/s}$$

Thus speed of the wave in the spring = 40 m/s.

Example 8.9

Compute and compare the speed of sound in air and that in steel rod.

The speed of a longitudinal wave in a steel rod is given by

$$\vartheta = \sqrt{\frac{Y}{\rho}}$$

where Y = Young's modulus and for steel $Y = 20 \times 10^{10}\ Nm^{-2}$

density of steel = $\rho = 7.8 \times 10^{3}\ kgm^{-3}$

therfore

$$\vartheta = \sqrt{\frac{20 \times 10^{10}\ (Nm^{-2})}{7.8 \times 10^{3}\ (kgm^{-3})}} = 5.06 \times 10^{3}\ ms^{-1}$$

The speed of sound in air is given by

$$\vartheta = \sqrt{\frac{r B}{\rho}}$$

where B = Bulk modulus and for air

$B = 1.01 \times 10^{5}\ Nm^{-2}$ and density of air ρ.

$\rho = 1.29\ kg\ m^{-3}$ $\gamma = 1.41$

therfore

$$\vartheta = \sqrt{\frac{1.01 \times 10^{5}\ (Nm^{-2}) \times 1.41}{1.29\ (kgm^{-3})}} = 3.2 \times 10^{3}\ ms^{-1}$$

Example 8.10

Compute the wavelength of the wave transmitted by a radio station that broadcasts on an assigned frequency of 1000 KHz.

All electromagnetic, including light, radio waves, television, etc travel with a speed of $3 \times 10^{8}\ ms^{-1}$ which is expressed in terms of wavelength and frequency by $\vartheta = \upsilon \lambda$

therefore

$$\lambda = \frac{\vartheta}{\upsilon} = \frac{3 \times 10^{8}\ ms^{-1}}{1000\ kHz} = \frac{3 \times 10^{8}\ ms^{-1}}{10^{6}\ Hz} = 300m$$

$\lambda = 300m$

Example 8.11

A note of frequency 500 Hz is emitted from an ambulance. (a) If the ambulance moves toward you at a speed of 60 km/h, what is

the frequency you will detect? (b) After the ambulance passes away from you its speed is 60 km/h. what frequency will you detect? The speed of sound is 340 ms^{-1}.

The speed of ambulance = 60 km/h = 16.7 m/s

(a) The frequency heard by the stationary listener is given by Eq. 8.37

$$\upsilon' = \frac{\vartheta\,\upsilon}{\vartheta - \vartheta_s}$$

where ϑ_s =16.7 ms^{-1}, υ = 500Hz and the velocity of sound ϑ = 340 ms^{-1}

therefore

$$\vartheta' = \left(\frac{340\ ms^{-1}}{340\ ms^{-1} - 16.7\ ms^{-1}}\right) 500Hz = 525.8\ Hz$$

(b) When the ambulance is moving away, the frequency heard by you is

$$\upsilon' = \frac{\vartheta\ \upsilon}{\vartheta + \vartheta_s} = \left(\frac{340\ m/s}{340\ m/s + 16.7\ m/s}\right) 500Hz$$

$$= 476.6\ Hz$$

Problems

1. An object is connected to one end of a horizontal spring whose other end is fixed. The object is pulled to the right (in the positive x-direction) by an externally applied force of magnitude 20N causing the spring to stretch through a displacement of 1 cm (a) Determine the value of force constant if, the mass of the object is 4kg(b)Determine the period of oscillation when the applied force is suddenly removed.

Ans (a) 2×10^3 N/m.
(b) 0.28108s

2. A body hanging from a spring is set into motion and the peri-

od of oscillation is found to be 0.50 s. After the body has come to rest, it is removed. How much shorter will the spring be when it comes to rest?

(Ans. 6.21 cm)

3. A pipe has a length of 2.46m.(a) Determine the frequencies of the fundamental mode and the first two overtones if the pipe is open at both ends. Take $\vartheta = 344$ m/s as the speed of sound in air. (b) What are the frequencies determined in (a) if the pipe is closed at one end? (c) For the case of open pipe, how many harmonics are present in the normal human being hearing range (20 to 20000 Hz)?

Ans. (a) 70 Hz, 140 Hz, 210 Hz
(b) 35 Hz, 105 Hz, 175 Hz
(c) 285 harmonics.

4. A standing wave is established in a 120 cm long string fixed at both ends. The string vibrates in four segments when driven at 120 Hz (a) Determine the wavelength (b) What is the fundamental frequency?

Ans. (a) 0.60 m
(b) 30Hz

5. Calculate the speed of sound in air at atmospheric pressure $p = 1.01 \times 10^5$ N/m^2, taking $\gamma = 1.40$ and $\rho = 1.2$ kg/m^3.

Ans. 343 m/s

6. A sound wave propagating in air has a frequency of 4000Hz. Calculate the precent change in wave length when the wave front, initially in a region where T = 27°C, enters a region where the air temperature decreases to 10°C.

Ans. 2.87%

7. The frequency of the second harmonic of an open pipe (open at both ends) is equal to the frequency of the second harmonic of a closed pipe (open at one end). (a) Find the ratio of the length of the closed pipe to the length of the open pipe. (b) If the fundamental frequency of the open pipe is 256Hz, what is the length of pipe? (Use $\vartheta = 340$ m/s)/

Ans. (a) 3/4

(b) $L_{open} = 0.664m$

$L_{closed} = 0.498m$

8. A 256 Hz tuning fork produces four beats per second when sounded with another fork of unknown frequency. What are two possible values for the unknown frequency?

Ans. 252Hz, 260Hz.

9. An ambulance travels down a highway at a speed of 75 mi/h. Its siren emits sound at a frequency of 400Hz. What is the frequency heard by a person in a car travelling at 55 mi/h in the opposite direction as the car approaches the ambulance and as the car moves away from the ambulance.

Ans. 477Hz, 337Hz.

10. A car has siren sounding a 2 kHz tone. What frequency will be detected as stationary observer as the car approaches him at 80 km/h? Speed of sound = 1200 km/h.

Ans. 2143 Hz

Chapter: 9

Nature of Light

9.1 DUAL NATURE OF LIGHT

The properties and the nature of light was studied by many scientists until about the middle of the seventeenth century. It was generally believed that light consists of a stream of tiny particles or corpuscles. The theory seemed to fit the facts of reflection, transmission and absorption of light. Newton in particular, supported the corpuscular theory of light. The theory suggests that light beam was attracted by the molecules of a denser medium which increased their velocity i.e. higher velocity in denser medium than the velocity in rarer medium.

In 1676, Christien Huygen (1629-1695), showed that a wave theory of light could explain the laws of reflection, refraction and the phenomena of double refraction. Two considerations discouraged Newton and others from accepting Huygen's wave theory. Firstly sound and water waves could bend round obstacles whereas light appeared not to do so, since it produced sharp shadows. Secondly, sound and water waves require a medium for their propagation. How, therefore, could a wave travel through vacuum as light must do to come from the sun and stars? The idea of a wave propagation without medium was unacceptable. One important difference between the theories was that the corpuscular theory of light predicated that light would travel faster in a material medium than air. Whereas the wave theory predicts a slower velocity in a material medium. The velocity of light could not at that time be measured in material, and the decision between the two theories was made on different evidence.

The first clear demonstration that light is a wave phenomena was made by the English physicist, Thomas Young (1773-1829), early in the 19th century. He demonstrated that under certain appropriate conditions, light exhibits an interference behaviour.

Had Newton been alive at the time, he would have appreciated the significance of Young's experiment. Unfortunately, Newton's name had become associated with the corpuscular theory, and admirers of Newton blindly supported the corpuscular theory and ignored the newly proposed theory of light. The wave theory became widely accepted in 1880. The first measurements of the velocity of light in the medium confirmed that the velocity of light is less than in a material medium than in air.Additional developments,such as interference and diffraction of light led to the general acceptance of the wave nature of light.

In secondary classes, the electrostatic field of a distribution of charges at rest and the magnetic field of a steady current in conductor have been studied. These fields may vary from point to point in space, but do not change with time at any individual point. For this situation these fields are regarded independent.

In case, the electric field and the magnetic field do vary with time, then it is not possible to treat the fields independently. According to Faraday's law changing magnetic field with respect to time acts as a source of electric field, similarly a changing electric field with respect to time acts as a source of magnetic field. Therefore, when magnetic field is changing with time, an oscillating electric field is induced in adjacent region of space and when an electric field is changing with time, an oscillating magnetic field is induced.

Consequently, an electromagnetic disturbance consisting of a time varying magnetic field and electric field can be conceived as electromagnetic waves which can propagate through space from one point to another. These paired fields are called collectively the electromagnetic field.

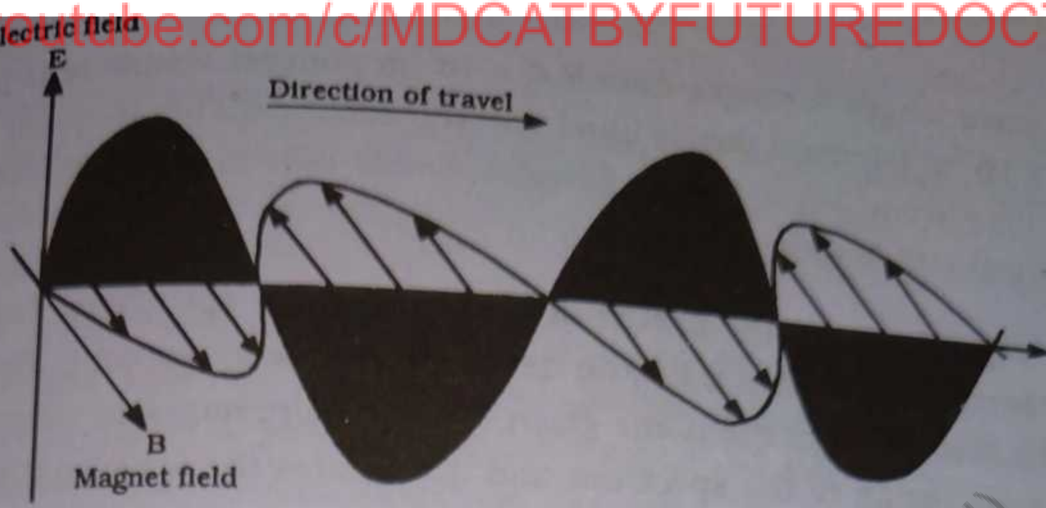


Fig. 9.1 At a large distance from the antenna (located in the negative x direction), the variation of E and B with distance at a given instant is, sinu soidal. Furthermore, the electromagnetic wave is a plane wave and has $E \perp B$ *at every point and for all value of the time.*

In1873,James Clark Maxwell showed that these electtromagnetic waves can propagate through vacuum. He obtained a theoretical expression for the propagation speed of the electromagnetic waves. The theoretical value of the speed of electromagnetic waves was found to be equal to the speed of light measured expermentally ($c = 2.998 \times 10^8\ ms^{-1}$).

Thus Maxwell concluded that light waves are electromagnetic waves. (This can also be expressed by stating that light propagates as an electromagnetic wave). These light waves consist of an oscillating electric field E and an oscillating magnetic field B. Both fields are perpendicular to each other. The mutually perpendicular magnetic and electric fields in the light wave oscillate in unison that is, same frequency and identical phase. Both E and B are perpendicular to the direction of propagation of the light waves which is along the x-axis as shown in Fig 9.1. The strength of the electric field at the same location and time is equal to the strength of magnetic field, multiplied by the speed of the light.

Finally, visible light is defined as radiation which can affect the human eye. For the particularly important case of visible light.

the wave length λ ranges from 7.6×10^{-7}m (longest visible red) to 4.0×10^{-7}m (shortest visible blue) and the corresponding frequency υ ranges from 3.9×10^{14}Hz (longest visible red) to 7.5×10^{14}Hz (shortest visible blue)

Fig 9.2 shows the sketch of electromagnetic spectrum. The electromagnetic spectrum has no definite upper or lower limit. The Fig 9.2 also displays the name given to the electromagnetic waves in various region of the spectrum and it indicates the wavelengths

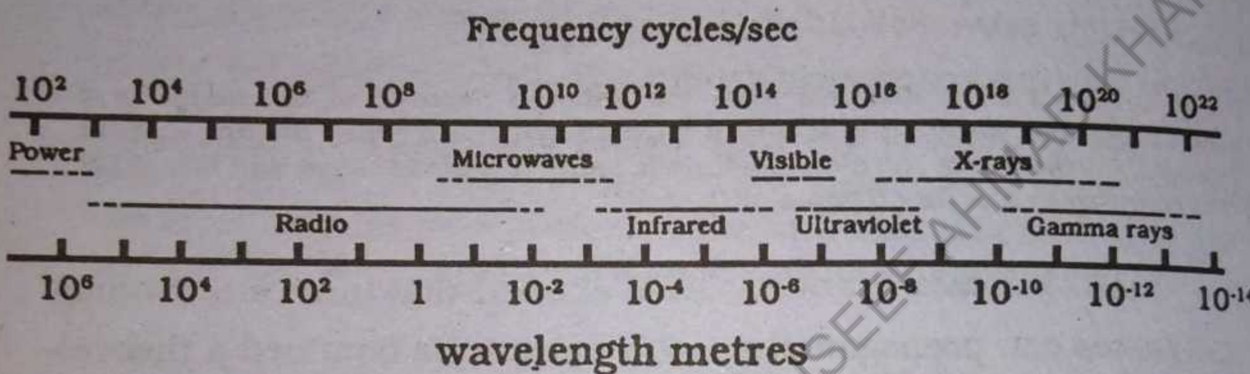


Fig. 9.2 The electromagnetic spectrum. Note that the wavelength and frequency scales are logarithmic.

and frequencies characterizing each. Taken together they constitute electromagnetic radiation.

The wave Theory of light, however, failed to explain the major features of photoelectric effect in which the ejection of electrons from a metal surface takes place when its surface is exposed to light.The experiment showed that.

(i) the Kinetic energy of the ejected photo electrons is independent of the light intensity while the wave theory suggests that Kinetic energy of the photo electrons depends upon the intensity of the light beam.

(ii) For each surface, there exists a cut off frequency. For frequencies less than the cut off frequency, there is no photoelectric effect, whereas according to the wave theory of light, the photoelectric effect should occur for any frequency of the light provided that light intensity is enough.

(iii) No detectable time lag has ever been observed between the impinging of the light on the surface and ejection of the photo electrons, whereas according to wave theory of light there should be a measurable time lag.

In 1905, Albert Einstein successfully explained the photo electric effect by making a remarkable assumption. His theory meets the three objections raised against the interpretation of the photo electric effect based on the wave theory of light. Using Planck's hypotheses of quantized energy exchanged. He proposed that all electromagnetic radiation was in the form of discrete bundles of electromagnetic energy, called photons. Each photon has total energy.

$$E = h\upsilon$$

Where h is Planck's constant ($h = 6.626 \times 10^{-34}$ J-s) and υ is frequency of the electromagnetic radiation. He further proposed that when a photon interacts with matter, it behaves as a praticle and delivers its entire energy to the individual electron in the absorbing surface. In 1921, he received the Nobel Prize in physics for his theory of photoelectric effect.

In 1921, photon nature of light was confirmed by A.H. Compton, which is called compton effect. He was successful in evaluating the motion of a single electron and a x-ray photon before and after collision between them and concluded that they behaved like material bodies and possess momentum and kinetic energy.

Millikan who experimentally verified the Einstein theory of photoelectric effect was also awarded the Nobel prize in 1923. These are several experiments which demonstrate without any ambiguity that light is composed of photons. The most straight forward photon correlation experiment was reported by Cluster in 1974.

Thus the photoelectric effect and compton effect supported the corpuscular theory of the light.

To summarize the discussion on the nature of light, we have seen that some phenomena such as interference, diffraction and polarization can be explained on the basis of wave theory of light. Whereas the photo electric effect and compton effect supported the corpuscular theory of the light. Consequently in the process of emission and absorption (to investigate the composition of radiation) it is observed that light radiation have particle- like properties. On the other hand in the phenomena of light propagation (to investigate where the radiation is) it is observed that radiation have wave-like properties. Thus discussion about the nature of light shows that light possesses dual nature i.e. wave-like and particle-like properties.

9.2 WAVE FRONTS, HUYGEN'S PRINCIPLE

In section 9.1, it has been shown that light waves are electromagnetic waves. These light waves consist of mutually perpendicular oscillating electric and magnetic fields. Both electric and magnetic fields are perpendicular to the direction of propagation, therefore, light waves are transverse waves. Waves can be classified as, one dimensional, two dimensional and three dimensional waves depending upon the number of dimensions in which they propagate energy.

Light waves which emanate radially from a small source are three dimensional. Consider a 'single wave', we can draw a surface through all points undergoing a similar disturbance at a given instant. As time passes, this surface moves along indicating how the wave propagates. For a periodic wave train we draw surfaces, all of whose points are in the same phase of motion. These surfaces are called 'wavefronts'. In homogeneous and isotropic medium, the direction of propagation is always perpendicular to the wavefront. A line normal to the wavefronts, indicating the direction of motion of the waves, is called a ray.

Wavefronts can take different shapes. For example, if the disturbances are propagated in a single direction, the waves are then referred as plane waves. Consider a plane normal to the direction

of propagation, here at a given instant conditions are the same everywhere on it. Such wavefronts are called plane wavefronts and the rays are parallel straight lines as shown in Fig 9.3. (a) If the distur-

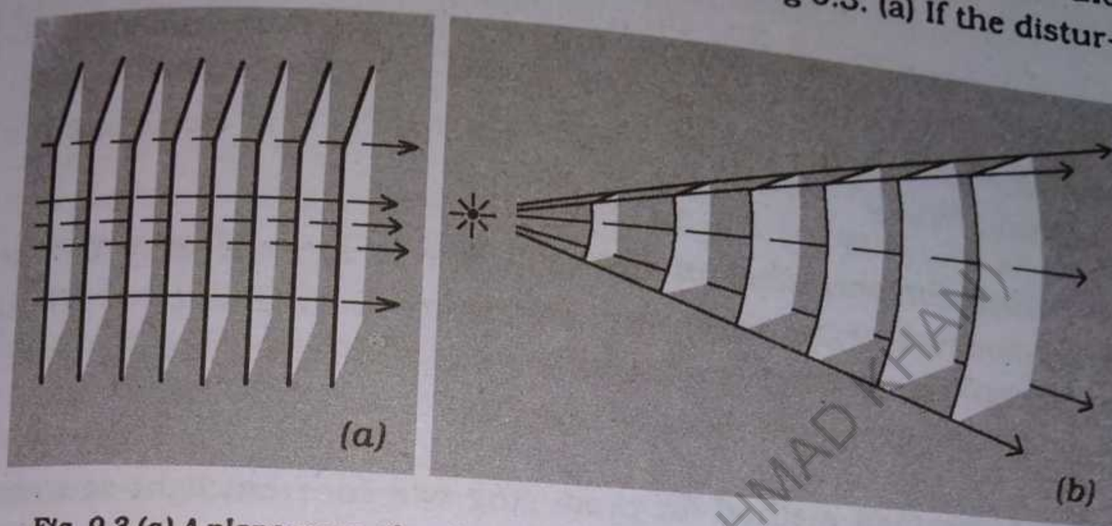


Fig. 9.3 (a) A plane wave. the planes represent wavefronts spaced a wavelength apart, and the represent rays. (b) spherical wave. The rays radial and the wavefronts, spaced a wavelength apart, form spherical shells. Far out from the source, however, small of the wavefronts become nearly plane.

bance is propagated outward in all possible directions from a point source, the resulting wavefronts are spheres and referred as spherical wavefronts while the rays are radial lines leaving the point source in all directions as shown in Fig 9.3(b). At a very large distance from the point source, the spherical wavefronts have very small curvature then a small portion of the spherical wavefront can be treated as plane, i.e. the spherical wavefront reduces to a plane wavefront under certain conditions.

9.3 INTERFERENCE OF LIGHT

In chapter 8, we have studied interference of sound waves which is the result of superposition of two waves. The interference phenomenon of waves, is a general feature of all types of waves such as sound waves, mechanical waves, light waves, etc.

Suppose two waves are allowed to superpose upon each other, if the resultant intensity of the interfering waves is zero or less than the intensity of the individual wave, then this type of interference is called destructive interference. If the resultant intensity of

the interfering waves is greater than the intensity of an individual wave then this type of interference is known as constructive interference.

Light waves also undergo interference. Fundamentally, all interference associated with light waves arises as a result of the combining of the electric and magnetic field vectors that constitute the individual wave.

Interference effect in the light waves are not easy to observe because of the short wavelengths involved (about 4×10^{-7}m to 7×10^{-7}m). In order to observe stable interference of light wave, the following condition must be obeyed:

A common method for producing two coherent light sources is to use one monochromatic source to illuminate a screen with two small slits as shown in Fig 9.4. The light emerging from both slits is coherent because a single source produces the original light

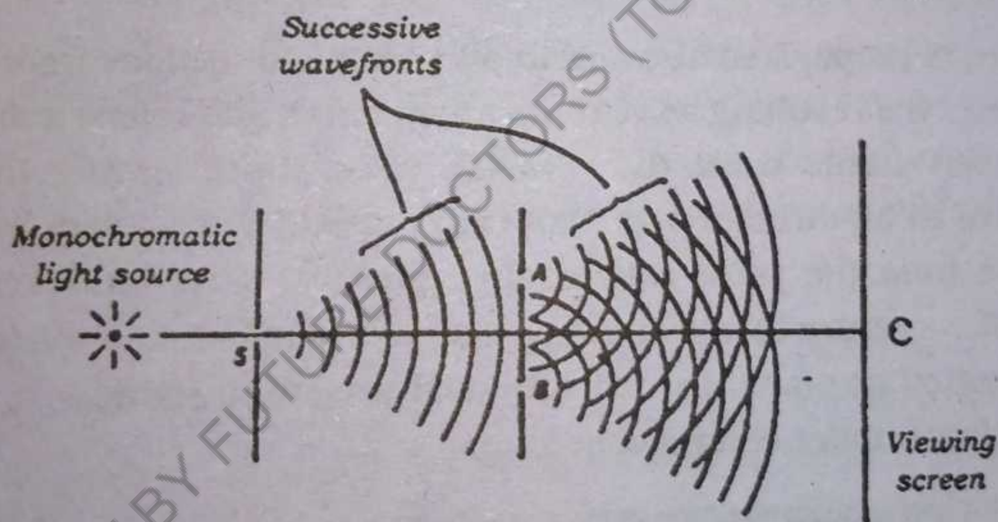


Fig. 9.4 Huyger.'; Construction of Interference pattern

beam and the two slits serve only to separate the original beam into the parts. Consequently, a random change in the light emitted by the source will occur in the two separate beams at the same time, and interference effects can be observed.

9.4 YOUNG'S DOUBLE-SLIT INTERFERENCE

The phenomena of interference in light waves from two sources was first demonstrated by Thoms Young in 1801. A sche-

matic diagram of the apparatus used by him during demonstration is shown in Fig 9.5(a). To obtain two coherent light sources light is incident on a screen, which has a narrow slit S_0. The waves emerging from this slit are then allowed to incident on a second screen which has two narrow, parallel slits S_1 and S_2. These two slits serve as a pair of coherent light sources because waves coming out from these slits originate from the same wavefront and therefore, always in phase. On a screen placed at some distance away. Young found a series of alternately dark and bright parallel bands corresponding to the position of destructive and constructive interference. These alternate dark and bright parallel bands are called fringes. That is, when two light waves add constructively at any location on the screen, a bright fringe is produced and when two light waves add destructively at any location on the screen a dark fringe is produced.

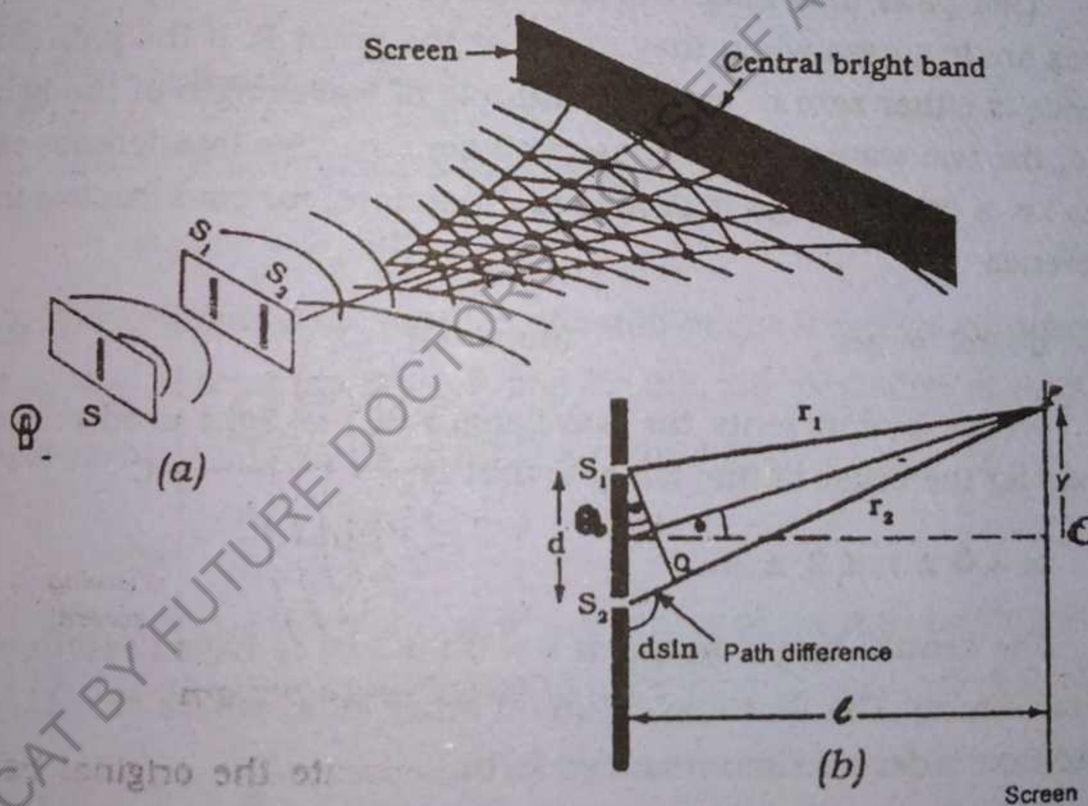


Fig. 9.5 (a) Young's two slit interference pattern (b) A diagram of the double slit experiment.

The quantitative description of Young's experiment can be obtained with the help of Fig 9.5 (b). Light waves with a definite wave length, λ, are incident on the pair of narrow, slits S_1 and S_2, which

are separated by a distance d. The interference pattern is observed on a screen which is placed at a perpendicular distance L from the screen containing slits S_1 and S_2, as shown in Fig 9.5 (b). Consider a point, P, on the viewing screen, suppose.

$$PS_1 = r_1 \text{ and } PS_2 = r_2$$

The light intensity on the screen at the point P is the resultant of the light coming from both slits. Note that a wave coming from the lower slit S_2 travels a greater distance than a wave from the upper slit by an amount equal to the difference between the two paths. the difference between PS_2 and PS_1 is known as path difference. from geometry in Fig. 9.5 (b), we get.

$$PS_2 - PS_1 = (r_2 - r_1) = d\sin\theta \qquad 9.1$$

This path difference will determine whether or not the two waves are in phase when they arrive at the point P. If the path difference is either zero or integral multiple of wavelength of the light used, the two waves are in phase and constructive interference results i.e. a bright fringe is produced. Therefore, for constructive interference

$$d\sin\theta = m\lambda \qquad \text{(maxima)} \qquad 9.2$$

Where λ represents the wavelength of the light used and m stands for the order of the fringes, that is

$$m = 0 \pm 1, \pm 2, \pm 3, \ldots$$

The central bright fringe at $\theta = 0$ ($m = 0$) is called zeroth order maximum. The first maximum on other side, where $m = \pm 1$, is called first order maximum and so forth.

Similarly, if the distance $(r_2 - r_1)$ contains an odd number of half wavelength (i.e. an integral number of wavelength plus one-half wavelength), the waves will arrive at the point P with their maxima displaced from one another by half wavelength ($\frac{1}{2}\lambda$). Therefore, at the point P the waves will be out of phase and destructive interfer-

ence will occur. Thus for destructive interference we have,

$$d \sin\theta = (m + \frac{1}{2})\lambda \qquad \text{(minima)} \qquad 9.3$$

Where $m = 0, \pm 1, \pm 2, \pm 3, \ldots$

Positions of the bright and dark fringes

It is useful to obtain expressions for the positions of the bright and dark fringes measured vertically from O to P. We shall assume that the distance from the slit to the screen is much larger than the distance between the two slits ($d \ll L$). In practice L is of the order of 1m while d is a fraction of a millimetre under these conditions θ is small, therefore

$$\sin\theta \equiv \tan\theta$$

In Fig 9.5(b) consider the triangle OPQ we see

$$\sin\theta = \tan\theta = \frac{Y}{L} \qquad 9.4$$

multiplying both sides by d we get

$$d \sin\theta = \frac{Y}{L} d \qquad 9.5$$

For computing the position of a mth bright fringe we substitute $y = y_m$ and comparing Eq. 9.5 and Eq.9.2, the positions of bright fringes measured from the point O are given by

$$y_m \times \frac{d}{L} = m\lambda$$

Where y_m be the distance of the centre of the mth bright band from the centre of the central band at $\theta = 0$

$$y_m = \frac{\lambda L}{d} m \qquad 9.6$$

Similarly using Eq. 9.3, and Eq. 9.5, we find that the dark fringes are located at

$$y_d = \frac{\lambda L}{d}(m + \frac{1}{2}) \qquad 9.7$$

From Eq. 9.6. we can calculate the distance between two adjacent bright and dark fringes. This distance is known as fringe spacing, Δx. As m increases by unit, we get

$$\text{fringe spacing} = \Delta x = \frac{\lambda L}{d} \qquad 9.8$$

As the quantities L and d are measurable, the wavelength of light used can be calculated. In fact Young used this method to obtain the wavelength of light. Additionally, the experiment gave the wave model of light a great deal of credibility.

9.5 INTERFERENCE IN THIN FILMS

Soap bubbles and thin layer of oil floating on water are common examples of thin films. When light is reflected from such a film, we often observe the vivid colours. These visible colour bands are the result of an interference effect. The interference in this case is caused by the interference of light waves reflected from the opposite surfaces of the thin films. We know white light consists of seven colours hence different wavelengths. Therefore the interference may be constructive or destructive depending on the phase relationship of the two interfering beams; hence the appearance of coloured fringes

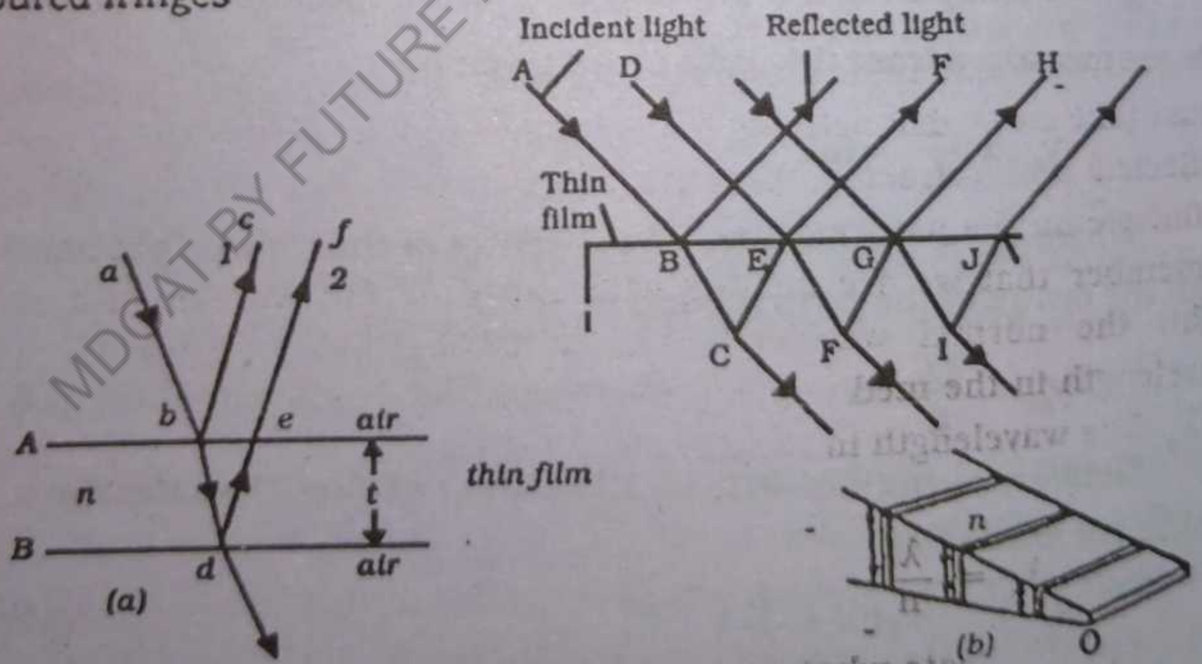


Fig. 9.6 (a) Interference in light reflected from thin film is due to a combination of rays reflected from the upper and lower surfaces
(b) Inter-ference produced by an air wedge.

To investigate the interference effect in thin film, we consider two plates of ultrathin glass plates separated by a thin uniform film of air as shown in fig 9.6(a). The film has a refractive index n and is surrounded by air whos refractive index is 1. We also assume that light waves are coming from a small part of an extended monochromatic source of light (thus relatively coherent) and traveling in air nearly perpendicular to the surface of the film.

In Fig 9.6(a), a line ab represents a ray in a beam of light shining on the upper surface of the film. At the point b on the air-glass interface, the ray ab is partially reflected as a ray be and partially transmitted as a ray bd. On the glass-air interface at the point d, another partial reflection occurs and some of the reflected light emerges as a ray ef. The ray be and ray ef interfere with each other; producing constructive or destructive interference depending on phase relationship. For nearly normal incidence the path difference between the two interfering rays (be and ef) is twice the thickness of the film. Thus path difference = 2t.

If this path difference is an integral multiple of wavelength, we expect constructive interference and if the path difference is a half integral number of wavelength destructive interference will occure.

Unfortunately, the situation is not quite simple. First, we must take care what actually happens to the phase of rays that are reflected and refracted, because these two effects have cirtical influence on the nature of the interference. Furthermore, we must remember that we are dealing with two different wavelengths of light; the normal wavelength λ in air and slightly different wavelength in the medium (of refractive index n) which is labelled as λ_n. The wavelength in medium is then given by

$$\lambda_n = \frac{\lambda}{n} \tag{9.9}$$

To investigate what happens to the phase of the rays. When the uniform thickness of the air film is made a thin film of variable thickness (a thin wedge of air). as shown in Fig 9.6(b). Along the

line where the plates are in contacts, there is no path difference and we expect a constructive interference (a bright spot) on the basis of Young's double slit experiment. But in actual experiment this does not happen. What happened? The straight forward conclusion to solve the problem is that one of the two rays has gone under a phase change of 180° during its reflection and therefore, the conditions for the production of constructive and destructive interference on their films are reversed.

In secondary classes you have studied that mechanical waves are reflected whenever they come to a fixed boundary. Observation shows that the reflected pulse is exactly out of phase (180° phase shift) with the original pulse. This phenomenon is as valid in the case of light waves as it is for mechanical waves; therefore the ray bc in Fig. 9.6 (a) which is reflected at the upper surface of the thin film is exactly out of phase (shifted by 180° or half a cycle) with the incoming ray. Refraction has no effect on the phase of the ray and is equivalent to pulse reflecting at the free end- an operation that does not affect the phase of the ray. Thus the ray ef emerges without any change of its phase. It is important to note that a light wave under goes a phase change of 180° upon reflection from a medium having an index of refraction greater than the index of refraction of the meduim in which the wave is travelling.

We conclude that only the ray bc which is reflected from the upper surface of the thin film undergoes phase reversal. The rays bc and ef which are out of phase, interfere with each other. By virtue of the phase shift of 180° (phase reversal) in the ray be, the condition for the production of constructive and destructive interference are reversed. Because the conditions for the production of constructive and destructive interference in case of Young's double slit experiments are based on the fact that there is no phase reversal in any interfering beam.

In the light of above discussion the condition for constructive interference can be expressed as

$$2t = (m + \frac{1}{2})\lambda_n; \; m = 0, 1, 2, 3$$

9.10

substituting. $\lambda_n = \frac{\lambda}{n}$ We get

$$2nt = (m + \frac{1}{2})\lambda; \; m = 0, 1, 2, 3, \ldots \text{(maxima)} \quad 9.11$$

and for destructive interference

$$2nt = m\lambda; \quad m = 0, 1, 2, 3, \ldots \text{(minima)} \quad 9.12$$

In practice, the interference fringes are not spaced equally in accordance with these equations because these conditions apply strictly only to the light that falls on the thin film at angles about 90° (nearly normal incidence). As remarked earlier the position of fringes depends upon critically on the wavelength of the light used, it is clear that the position for the bright and dark circles will be different for different wavelengths. Hence when white light is reflected and refracted from a thin film produces coloured fringes due to the difference in wavelengths of the colour that is make it up.

9.6 NEWTON'S RINGS

An air wedge may be formed between the curved surface of a plano convex lens and the plane surface of glass shown in Fig. (9.7).

This arrangement produces an interference pattern consisting of a number of rings centered on the point of contact between the lens and the plane surface. These rings are called Newton's rings. The thickness of the air film between the glass surfaces varies from zero at the point of contact to some value 't' at some point E. as shown in fig 9.7.

If the radius of curvature of lens R is very large as compared with the radius r (i.e radius of a ring). The point of contact gives a dark circle due to zero path difference at this point. and 180°

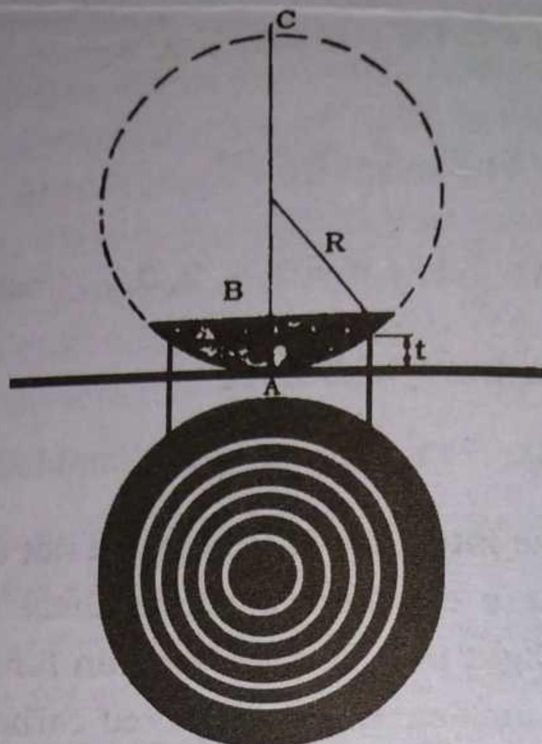


Fig. 9.7 Newton's rings produced by interference at the air wedge formed by the lower surface of the lens and a plane glass surface.

change in phase in the light externally reflected at the lower surface. The photograph of Newton's rings shows a series of dark and bright fringes. These are due to destructive and constructive interference using the geometrical theorem that the product of intercepts of intersecting chords are equal. we have

$$r^2 = (BC) \times (AB)$$

from Fig. 9.7, we rite

$$BC = (2R - t)\ ,\ AB = t$$

$$r^2 = (2R - t) \times (t) = 2Rt - t^2$$

t^2 being small is neglected, we get

$$r^2 = 2tR \qquad 9.13\,(a)$$

$$r = \sqrt{2tR} \qquad 9.13\,(b)$$

The path difference for constructive interference in this film is given by Eq. 9.11

$$2nt = \left(m + \frac{1}{2}\right)\lambda$$

assuming n =1 (for air)

$$2t = \left(m + \frac{1}{2}\right)\lambda$$

for first bright ring (m = 0), we write

$$2t_1 = \left(0 + \frac{1}{2}\right)\lambda = \frac{1}{2}\lambda$$

for second bright ring (m = 1), we write

$$2t_2 = \left(1 + \frac{1}{2}\right)\lambda = \frac{3}{2}\lambda$$

for third bright ring (m = 2)

$$2t_3 = \left(2 + \frac{1}{2}\right)\lambda = \frac{5}{2}\lambda$$

similarly, for Nth bright ring. (note that N = m+1) we write

$$2t_N = \left((N-1) + \frac{1}{2}\right)\lambda = \left(N - \frac{1}{2}\right)\lambda$$

Substituting $2t_N$ in eq: 9.13 (a) and 9.13 (b), we get

$$r_n^2 = R\left(N - \frac{1}{2}\right)\lambda \qquad 9.14\ (a)$$

$$r_n = \sqrt{R\left(N - \frac{1}{2}\right)\lambda} \qquad 9.14\ (b)$$

Example: 9.1

If the radius of the 14th Newton's ring is 1mm, when light of wavelength 5.89×10^{-7}m is used. What is the radius of curvature of the lower surface of the lens used?

Solution

$$r_N^2 = R\left(N - \frac{1}{2}\right)\lambda$$

r = 1mm

$N = 14$

$\lambda = 5.89 \times 10^{-4}$ mm

Therefore

$$l^2 = R\left(14 - \frac{1}{2}\right) \times 5.89 \times 10^{-4}$$

$$R = 125.7\text{mm}$$

9.7 THE MICHELSON INTERFEROMETER

The michelson interferometer was invented by an American physicist A.A. Michelson (1852-1931). The Michelson interferometer played an interesting role in the history of science during the latter part of the nineteenth century. It has a great scientific importance and had an equally important role in establishing high precision standards of the unit of length. In contrast to the Young's double slit experiment for producing interference fringes which make use of light from two narrow sources, the Michelson interferometer uses light from a broad, spread source (extended source).

A schematic diagram of the interferometer is shown in Fig 9.8.(a)

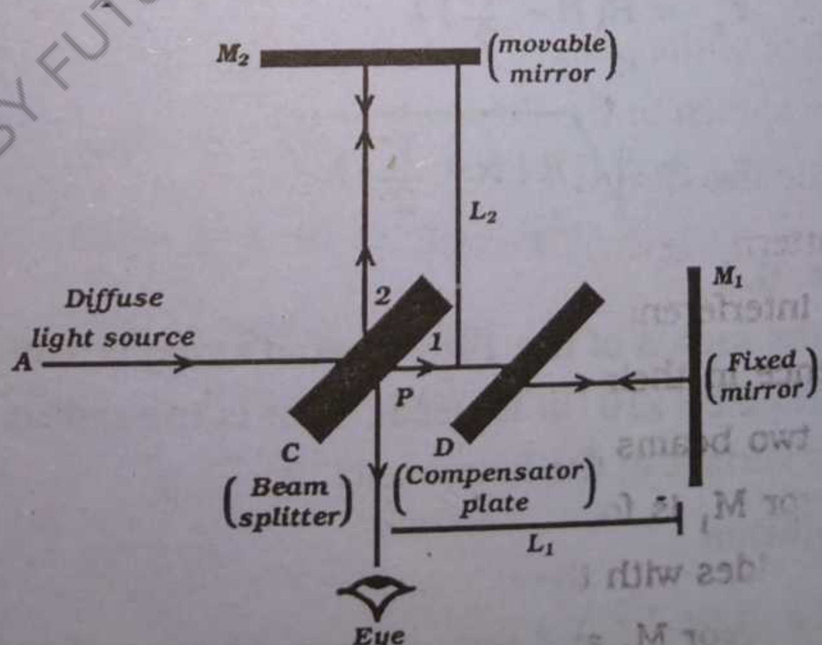


Fig. 9.8 (a) The Michelson interferometer.

It consists of two highly polished plane mirrors M_1 and M_2. The mirror M_1 is fixed where as the mirror M_2 is movable as shown in the diagram. In addition to this, it has a glass plate C which has a thin coating of silver on its right side. This partially silvered plate is called beam splitter and is inclined at angle of 45° relative to the incident light beam. It has also another plate D which is identical to the plate C except it is not silvered. Its purpose is to ensure that the beam 1 and 2 pass through the same thickness of glass as shown in Fig 9.8.(a) Therefore, it is known as compensating plate. This is particularly important when white light fringes (coloured fringes) are desired. A monochromatic beam of light from an extended source of light A falls on the half silvered plate C. Part of the light is reflected from the silver surface of the beam splitter C (at the point p) to the movable mirror M_2. After reflection at M_2 it returns to the observer's eye through the plate C. The remaining part of the light passes through silvered surface of the plate C, continues its journey, passes through the compensating plate D and finally falls on the fixed mirror M_1. The light is reflected back from the fixed mirror M_1. It passes through the compensating plate D on its return journey and finally it is incident on the silver surface of the plate C from where it is reflected to the observer's eye as shown in the Fig.9.8(a). After reflection from mirror M_1 and M_2, the two beams eventually recombine to produce an interference pattern which can be viewed.

The interference pattern for the two beams is determined by the difference in their path lengths L_1 and L_2 as shown in figure. When the two beams are viewed as shown, the virtual image (say M_1) of mirror M_1 is formed by reflection at the silvered surface of plate C coincides with the mirror M_2 provided L_1 is exactly equal to L_2 and the mirror M_1 and M_2 are kept exactly at right angles. If L_1 and L_2 are not exactly equal, the image of the mirror M_1 is displaced slightly from M_2 (still parallel to one another); and if the an-

gle between the mirrors is not exactly 90°, the virtual image M_1' of the mirror M_1 makes a slight angle with M_2. Under this situation, the mirror M_2 and the virtual image M_1 of the mirror M_1 behave in the same manner as the two surfaces of a thin film, and the same sort of interference fringes result from the light reflected from these surfaces. The effective thickness of the air film is varied by moving mirror M_2 parallel to itself. Under these conditions, the interference pattern is a series of bright and dark rings.If the extended source is monochromatic. If a dark ring appears at the centre of the interference pattern, the two beams interfere destructively. If a bright ring appears at the centre of the interference pattern, the two beams interfere constructively.

Suppose the extended source is monochromatic of wavelength λ, and the mirror M_2 is then moved a distance λ/4, the path difference changes by λ/2 (twice the separation between M_2 and M_1) a dark ring will appear again at the centre of the interference pattern. Thus, successive dark and bright rings are formed each time M_2 is moved a distance λ/4. The wavelength of light used is then measured by counting the number of fringes shift for a given displacement of the mirror M_2. If the displacement is represented by x, then

$$x = m\frac{\lambda}{2} \quad or \quad \lambda = \frac{2x}{m}$$

If m is several thousand, the displacement x is large enough so that it can be measured with good precision, and hence a precise value of the wavelength λ can be determined. A common type of Michelson interferometer used in laboratories is shown in Fig. 9.8 (b).

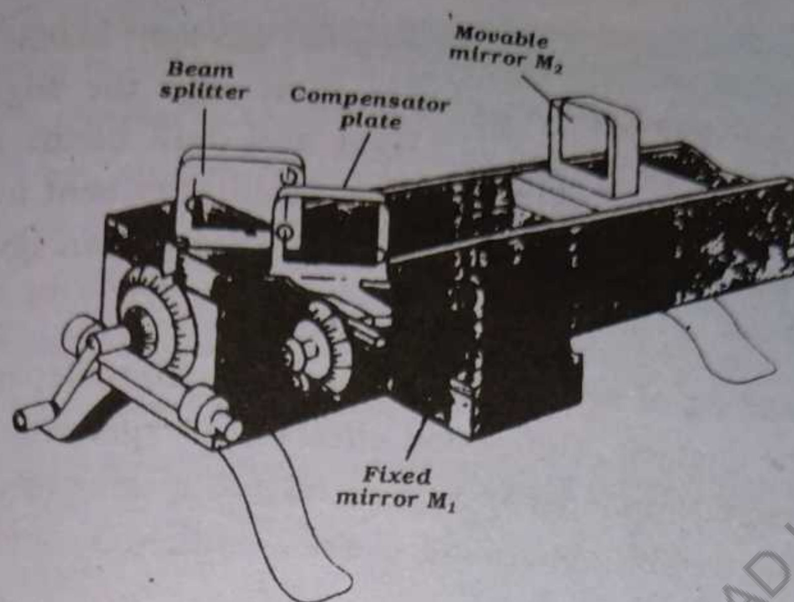


Fig. 9.8 (b) A common type of Michelson interferometer.

9.8 DIFFRACTION

According to the principles of geometrical optics, if we place an opaque object (an object through which light cannot pass) between a point source of light and a screen, a shadow of the obstacle is formed on the screen. In addition, we also observe the following :

(i) No light reaches within the geometrical shadow of the obstacle at the screen.

(ii) Outside the geometrical shadow the screen is uniformally illuminated.

Fig. 9.9 shows the shadow of a razor blade placed between a point source of a mono-chromatic light and a photographic plate.

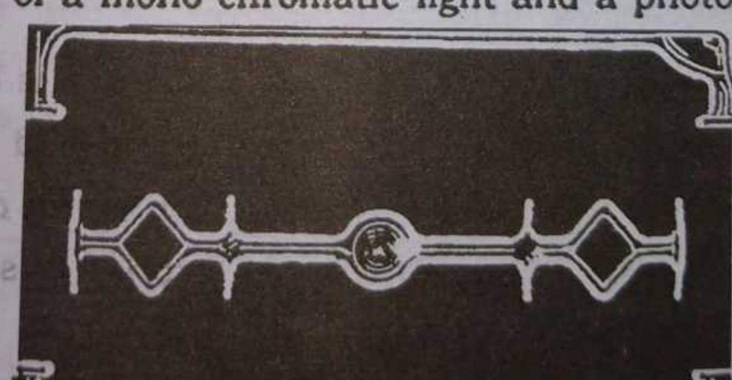


Fig. 9.9 The shadow produced when a razor blade is illuminated by monchromatic point source.

Contrary to what we have observed earlier (no light reaches within the geometrical shadow), we observe that near the edge of the shadow a pattern consisting of bright and dark bands appears. This leads to conclude that some of the light has bent inside the geometrical shadow. "The bending of light around an obstacle is called diffraction."

The bending of light i.e. the diffraction effect depends upon the size of the obstacle. Diffraction effects are larger only when we deal with obstacles or apertures comparable in size to the wave-length. Usually the diffraction effects are small and must be looked carefully.

The phenomenon of diffraction was discovered by Francesco Maria Grimaldi (1618-1663). The diffraction effect was known to Newton (1642-1727), but he did not see in it any justification for a wave theory of light. Huygens although believed in wave theory of light but did not believe in diffraction phenomenon in light. Fresnel (1788-1827) correctly applied Huygen's principle to explain the phenomenon of diffraction which could not be explained on the basis of ray optics.

Diffraction effects are classified in two types.

FRESNEL DIFFRACTION

When both the point source and screen at which the diffraction pattern is formed are kept at finite distance from the diffracting obstacle, the wavefronts falling on the obstacle are not plane; the corresponding rays are not parallel. Similarly the wavefronts leaving the aperture or obstacle to illuminate the screen are not plane as shown in Fig.9.10(a). This situation is described as Fresnel diffraction.

FRAUNHOFER DIFFRACTION

If the source and the screen on which the diffraction pattern is formed are removed at a large distance, so that the correspond-

ing rays are parallel to each other and the wavefronts are plane. This situation is described as Fraunhofer diffraction. This class of diffraction is simpler to treat analytically and can be established in laboratories by using two converging lenses. A lens between distant source of light and obstacle, renders the rays parallel to each other and hence produces plane wavefronts. Whereas second lens collects the parallel set of diffracted rays and focus them at a point on the screen as shown in Fig. 9.10(b and c)

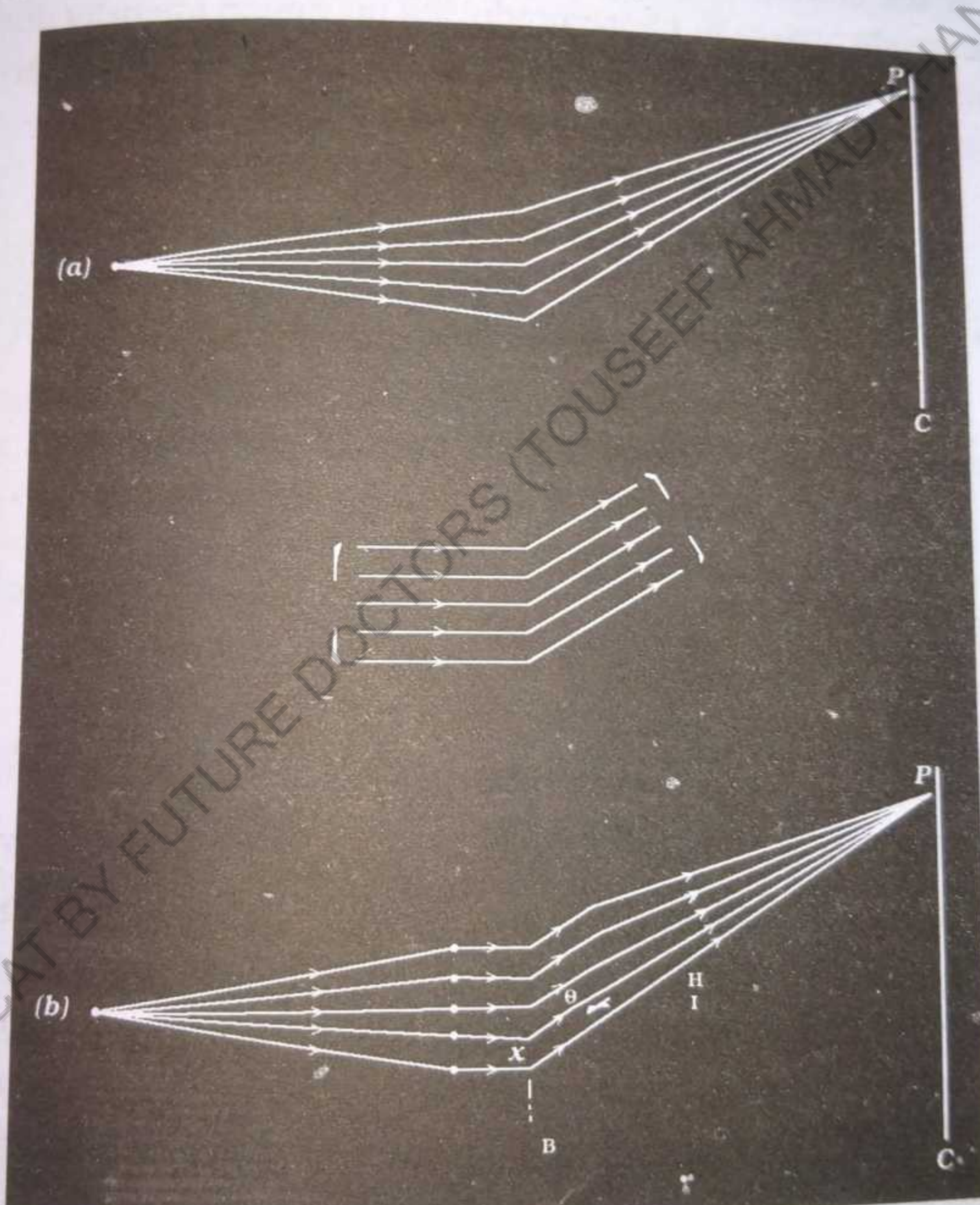


Fig. 9.10 (a) Fresnel diffraction. (b) Source S and screen C are moved to a large distance, resulting in Fraunhofer diffraction. (c) Fraunhofer diffraction conditions produced by lenses, leaving source S and screen C in their original positions.

Single Slit experiment

To understand how dark and bright bands appear in the diffraction pattern inside the geometrical shadow of an obstacle or aperture, we consider a single slit which is gradually narrowed in steps as shown in Fig. 9.11. As the slit gets narrower, the divergence of the diffracted light increases from the slit (i.e. bending around obstacle or aperture increases). In addition, it produces a pattern of light called diffraction pattern as remarked earlier. The centre of the diffraction pattern is bright since the light from all parts of the slit arrives in phase producing maximum intensity due to constructive interference. Single source (i.e. single slit) interference is referred as diffraction and occurs for all kinds of waves.

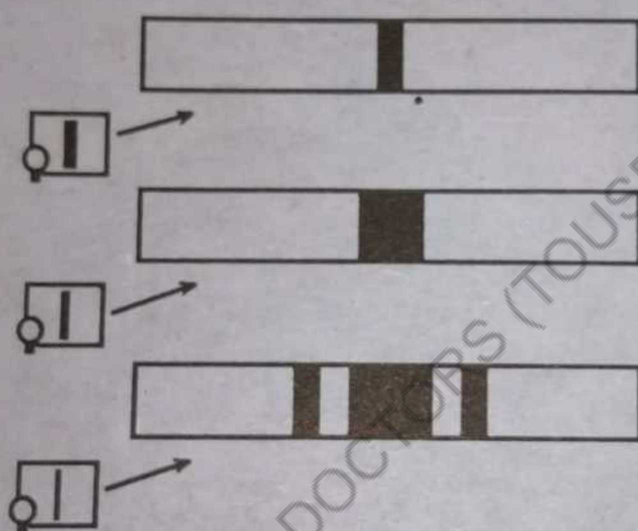


Fig. 9.11 Diffraction pattern of a slit showing the effect of reducing the width of the slit.

Consider a set of parallel rays fall on the slit of width a. The light bends around the edges as shown in Fig.9.12, we divide the slit into two halves. Mark of equidistant points above and below the centre of the slit such as c – c´, d– d´, e – e´, f – f´,... etc. Suppose the path difference (p.d.) between the rays from the two edges of the slits is one wavelength (1λ) as shown in Fig 9.12.. (ii)

Obviously the path difference between the rays from the centre point C and C from the upper edge of the slit is $\lambda/2$ (i.e. 180°). the two rays are out of phase and cancel each other. Similarly the

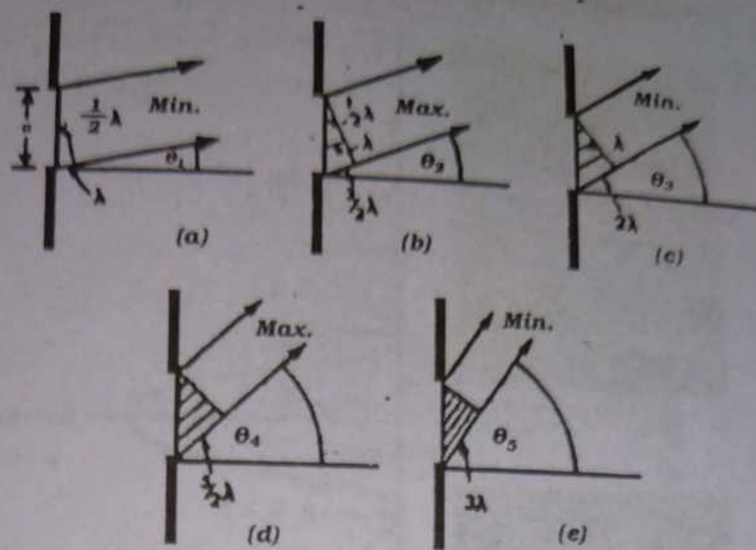


Fig. 9.12 We can obtain the directions of the maxima and minima in a diffraction pattern by dividing up the slit and considering it as a case of interference.

rays from the corresponding pairs of points d – d′, e – e′, f – f′, etc.... from the lower and upper half of the slit cancel in pairs and produces zero intensity in this particular direction. If the angle increases so that the p.d. between the edges of the slit is $\frac{3}{2}\lambda$, a peak in the intensity occurs.

This is best understood by dividing the slit into three equal parts as shown in Fig: 9.12(b).The light from the upper edge and the central parts interfere destructively, as seen earlier and produces zero intensity. Light from the lowest third part is transmitted uncancelled and results in producing a peak in intensity. Fig. 9.12 (c,d and e) show the path difference is 2λ, $\frac{5\lambda}{2}$, 3λ and the corresponding to each path difference, we get zero intensity, maximum intensity and zero intensity respectively.

Consequently,(using figure) we can write

$$a\sin\theta = 0, \pm\frac{3}{2}\lambda, \pm\frac{5}{2}\lambda, \pm\frac{7}{2}\lambda, \ldots \text{ (for diffraction maxima)} \quad 9.15$$

$$a\sin\theta = \pm\lambda, \pm 2\lambda, \pm 3\lambda, \pm\lambda, \ldots \text{ (for diffraction minima)} \quad 9.16\text{ (a)}$$

or $$a\sin\theta = m\lambda ; m = \pm 1, \pm 2, \pm 3, \ldots \quad 9.16\text{ (b)}$$

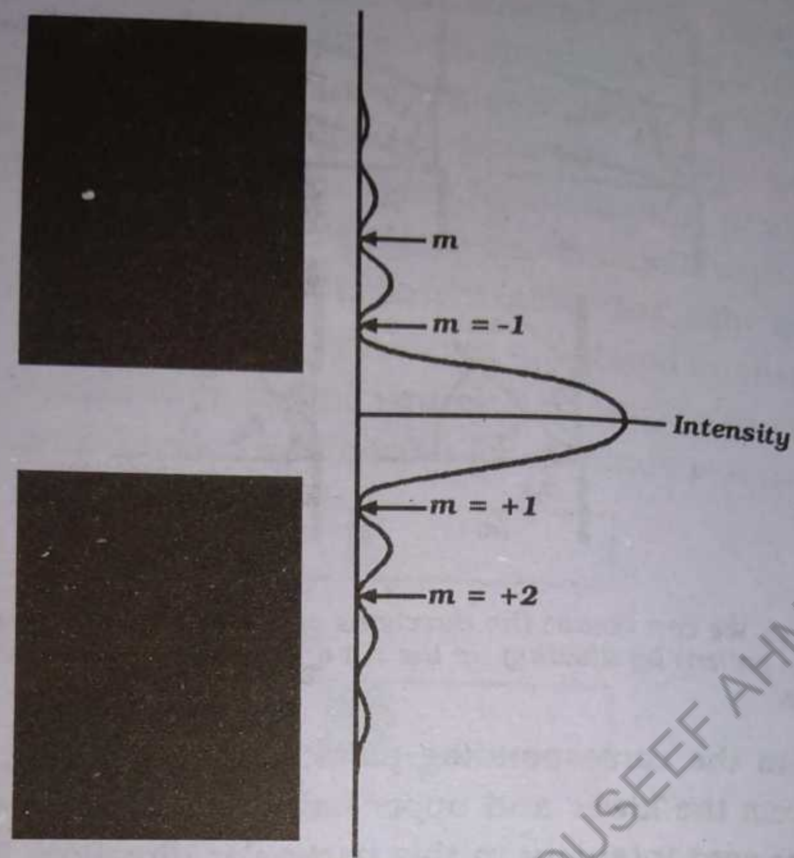


Fig 9.13 (a) Photograph of the diffraction pattern produced by a single slit illuminated by monochromatic light. (b) intensity graph for this diffraction pattern.

The central m=0 maximum is very bright since all the wavelets have nearly the same path length and are in phase. The diffraction pattern produced by single slit and the intensity graph for this diffraction pattern when illuminated by a monochromatic light is shown in Fig.9.13.

9.9 DIFFRACTION GRATING

Suppose that instead of a single slit or two slits side by side, we have a very large number of parallel slits, all with the same width and spaced equal distance apart such an arrangement is called a diffraction grating. By using large number of slits, the intensity and the sharpness of lines can be increased, enabling the wave length of the light to be accurately measured.

A diffraction grating is a very useful device for analysing light sources. A diffraction grating consists of a piece of glass with num-

ber of parallel lines marked on it. The thin clear strips between the lines transmit light and act as slits. A fine grating with 6000 lines per cm has a slit spacing d equal to 1.66×10^{-4} cm.

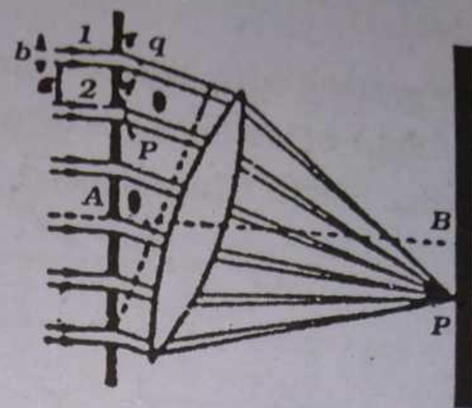


Fig. 9.14 A diffraction grating.

A schematic diagram of plane diffraction grating is shown in Fig: 9.14 where b is the width of slit and a is the separation between two consecutive slits. $(a + b) = d$ is called grating element. A parallel beam of light from the monochromatic source is falling on the grating, which sends out waves from each slit. A convex lens can be used to bring the waves together at a point along certain definite directions, waves of a particular wave length from adjacent slits are in phase and reinforce each other.

The parallel rays of light after diffraction through the grating if differ by one wave length when they arrive at p, they will interfere constructively. The condition for constructive interference is that the path difference between two consecutive waves should be equal to $d \sin\theta$ shown in Fig: (9.14).

For the central maximum there is no path difference, so we call it zero order.

The Ist order maximum occurs when

$d \sin\theta = \lambda$

The 2nd order maximum occurs when

$d \sin\theta = 2\lambda$

For the mth order maximum

$$d \sin\theta = m\lambda \; ; \; m = \pm 1, \pm 2, \pm 3$$

Example : 9.2

A diffraction grating with 10000 lines per centimetre is illuminated by yellow light of wavelength 589 nm. At what angles are the 1st and 2nd order bright fringes seen?

Solution

$$d \sin\theta = m\lambda$$

for first order m = 1

$$\sin\theta = m\lambda \backslash d = 1\,(589 \times 10^{-9}) \,/\, /10000 \times 1/\,100 = 0.589$$

$$\sin\theta = 36.1 \text{ degrees}$$

For second order, m = 2

$$\sin\theta = m\lambda \backslash d = 2\,(589 \times 10^{-9}) / (1/10000 \times 1/100) = 1.178$$

This number is larger than 1 and so sinθ does not represent any rational angle.

Hence only the 1st order appaers for this wavelength.

9.10 DIFFRACTION OF X-RAYS THROUGH CRYSTALS

X-rays are electromagnetic waves. They have short wave length of the order of 10×10^{-10} m, therefore it is not possible to produce interference fringes of X-rays by Young's double slit method or by thin film method. The reason is that the fringe spacing is given by $\frac{\lambda}{d} \cdot L$ and unless the slits are separated by a distance of the order of 10×10^{-10}m, the fringes obtained will be closed together that they cannot be observed.

However, it is possible to obtain x-rays differaction by making use of crystals such as rock salt in which the atoms are unifor-

mally spaced in planes and separated by a distance of the order of 2-5A°. Therefore, the diffraction of x-rays takes place when they incident on the crystal as shown in Fig: 9.15.

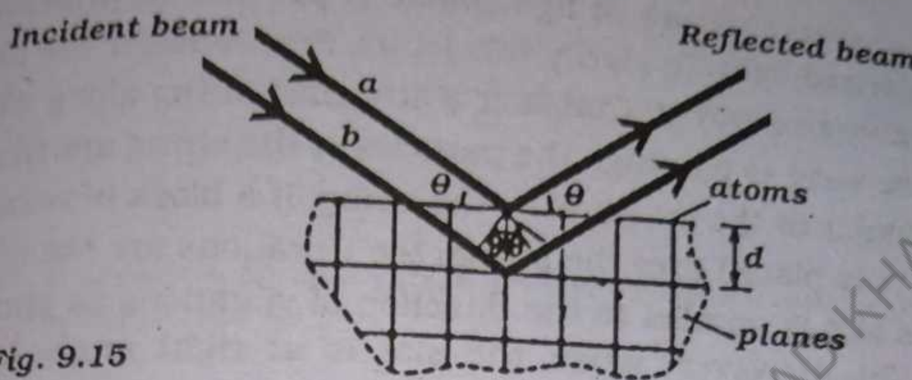


Fig. 9.15

Consider a set of parallel lattice planes having spacing d between each other shown in Fig. 9.15. Consider a and b rays reflected from the two layers separated by a distance d. The path difference between the two reflected rays. is. LB + BC (see Fig 9.15).

It can be proved

BC = LB = d sinθ , where d is distance between the atomic planes of the crystal. Therefore the path difference (LB+BC) = 2d sinθ. Now the waves will interfere constructively if the path difference is an integral multiple of the wave length.

$$m\lambda = 2d \sin\theta$$

This relation is called Bragg's Law. The spacing of the atomic layers of crystals can be found from the density and atomic weight. Both m and θ can be measured and hence the wave length of x-rays can be determined from the above equation. Conversely if λ is known, the d may be measured.

9.11 POLARIZATION OF LIGHT WAVES

From the phenomena of interference and diffraction of light it was proved that light has a wave nature. However, this does not tell about the type of waves.

There is a periodic fluctuation in electric and magnetic fields along the propagation of light wave. These fields vary at right angles to the direction of the propagation of the light wave, so light wave is transeverse wave.

Transverse nature of light make it possible to produce and detect polarized light. To clarify this let us, first consider the behaviour of transverse waves. Consider a stretched string along which a transverse wave is passing. The particles of the string are vibrating perpendicular to the length of the string. If a block of wood with a slot in it is placed over the string, the vibrations are not affected when the slot is parallel to the direction of vibrations as shown in fig.9.16 (b). However, when the slot is at right angle to this direction, the vibrations do not pass.

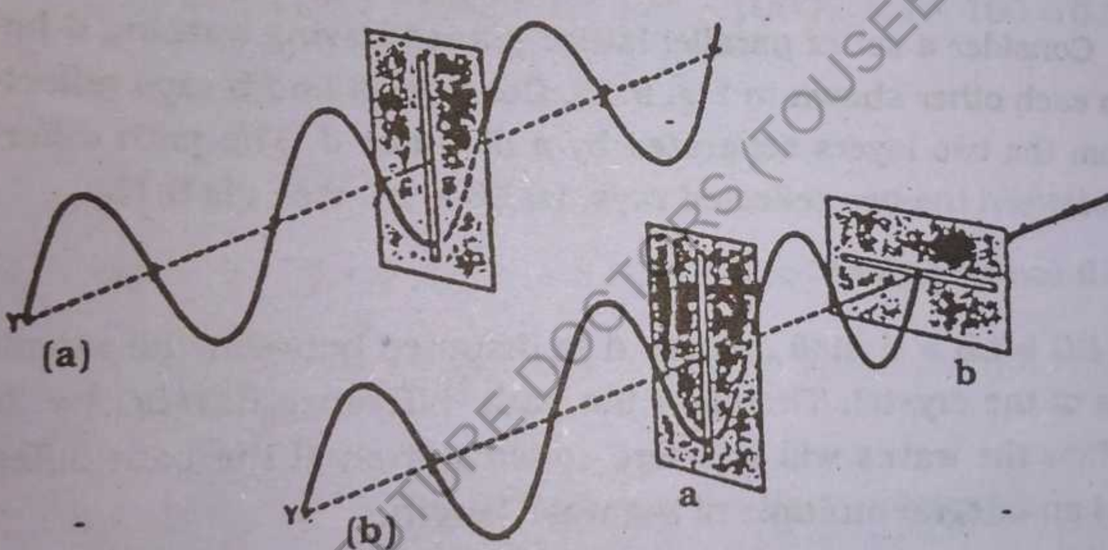


Fig: 9.16 (a)The transverse vibrations on a string are not affected if the slot is (a) parallel to the direction of vibration.
(b) The vibrations do not pass through the slot (b) if it is held perpendicular to the direction of vibration.

A beam of light from the normal source contains large number of waves. The direction of whose vibrations is completely different. This is shown in Fig: 9.17(a) .The beam of light is said to be polarized; if un polarized beam passes through a polarizing sheet known as polariod. Unpolarized can be represented as two linearly polarized beams at right angle to each other as shown in Fig: 9.17 (b and c)

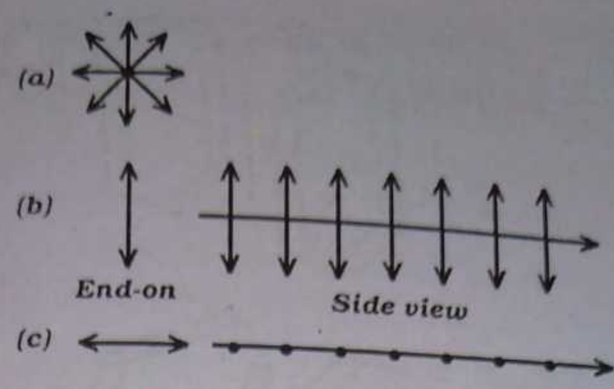


Fig 9.17 (a) unpolarized ordinary light. (b) plane polarized light (c) Polarized perpendicular to paper.

Polarization depends on a parallel arrangement of crystals, which has two effects on the light.

1. It resolves the direction of the vibration of the light wave in to only two directions mutually at right angles.
2. It absorbs one of these components and transmit the other as in Fig: 9.18.

The plane polarized light can be obtained by passing light through a tourmaline crystal. When two tourmaline crystals are placed parallel to each other the light transmitted by the first crystal is also transmitted by the second crystal. When the second crystal is rotated through 90 degrees, no light gets through. The observed effect is due to selective absorption by tourmaline of all light waves vibrating in one particular plane, the second crystal is known as analyser and the first crystal is known as polarizer as shown in Fig: 9.18.

The method of polarizing the light discussed above is called polarization by selective absorption. However, light can be polarized by other methods like reflection, doublerefraction and scattering of light.

Polarization of light has many technical and scientific applications in daily life. These include :

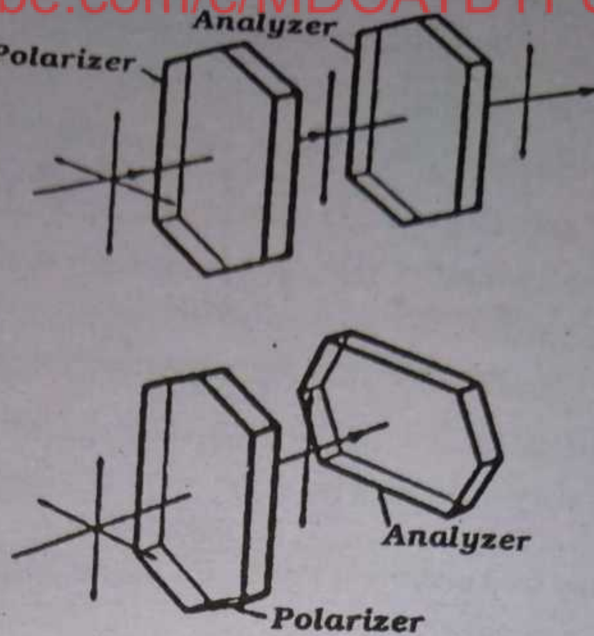


Figure: 9.18

1. The determination of the concentration of optically active substance such as sugar.

2. In photography it is often desirable to enhance the effect of sky and clouds. Since light from the blue sky is partially polarized by scattering a suitable orientated polarizing disc infront of the camera lens will serve as a sky filter.

QUESTIONS

1. Distinguish between diffraction and interference. Can there be diffraction without interference, and interference without diffraction ?

2. Describe and explain the interference effect produced by thin films an observer sees red colour at certain position in an oil film Would other observers also see the red colour at the same position?

3. Consider YOUNG'S double slit experiment and explain what the following parameters have to do with the light distribution on the screen.

   (a) Distance between the slits.

(b) Width of the slits.

(c) Wavelength of the incident light.

4. Give an experimental arrangement for producing Newton's rings. Why are the fringes circular and why is central spot black? Would the point of central spot appears dark if seen with the help of transmitted light?

5. Discuss the statement that a diffraction grating could just as will be called an interference grating?

6. For a given family of planes in a crystal, can the wavelength of incident X-rays be:

   (a) too large or

   (b) too small to form a diffracted beam?

7. What do you mean by plane polarized light? How does the phenomena decide that light waves are transverse?

8. Why Maxwells discovery that light was an electromagnetic wave is so important?

9. Which of the following can occur in (a) transverse and (b) longitudinal waves?

   Refraction, Dispersion, Interference, Diffraction, and Polarization.

PROBLEMS

1. How many fringes will pass a reference point if the mirror of a Michelson's interferometer is moved by 0.08mm. The wavelength of light used is 5800A.

   (Ans. 275)

2. In a double slit experiment the separation of the slits is 1.9mm and the fringe spacing is 0.31mm at a distance of 1 from the slits. Find the wavelength of light?

   (Ans.-0.589×10^{-6} m)

3. Interference fringes were produced by two slits 0.25mm apart on a screen 150mm from the slits. If eight fringes occu-

by 2.62mm. What is the wavelength of the light producing the fringes?

(Ans. 5.46x10 m)

4. Green light of wavelength 5400A° is diffracted by gratting having 2000 lines cm.

(a) Compute the angular deviation of the third order image.

(b) Is a 10th order image possible?

Ans. (a)189° (b)Impossible.)

5. Light of a wavelength 6×10^{-7} m falls normally on a diffraction grating with 400 lines per mm. At what angle to the normal are the 1st, 2nd, and 3rd order spectra produced?

Ans. 13.9°, 28.7°, 46.1°)

6. If a diffraction grating produced a 1st order spectrum of light of wavelength 6×10^{-7} m at an angle of 20° from the normal. What is its spacing and also calculate the number of lines per mm?

(Ans: 1.75×10^{-3}mm, 5.7×10^{2} lines / mm)

7. Newton's rings are formed between a lens and a flat glass surface of wavelength 5.88×10^{-7} m. If the right passes through the gap at 30^{0} to the vertical and the fifth dark ring is of diameter 9mm. What is the radius of the curvature of the lens?

(Ans. 23.8m)

8. How far apart are the diffracting planes in a NaCl crystal for which X-rays of wavelength 1.54A° make a glancing angle of 15° – 54 in the 1st order?

(Ans. 2.81A°)

9. A parallel beam of X-rays is diffracted by rocksalt crystal the 1st order maximum being obtained when the glancing angle of incidence is 6 degree and 5 minutes. The distance between the atomic planes of the crystal is 2.81×10^{-10}m. Calculate the wavelength of the radiation.

(Ans. 0.5952×10^{-10}m)

Geometrical Optics

10.1 LENSES

A lens is a piece of transparent material that can focus a transmitted beam of light. This is usually bounded by two spherical surfaces, or a spherical and a plane surface. For our convenience, we will deal with thin lenses in this chapter. Basically lenses fall into two categories, converging or convex lenses, and diverging or concave lenses.

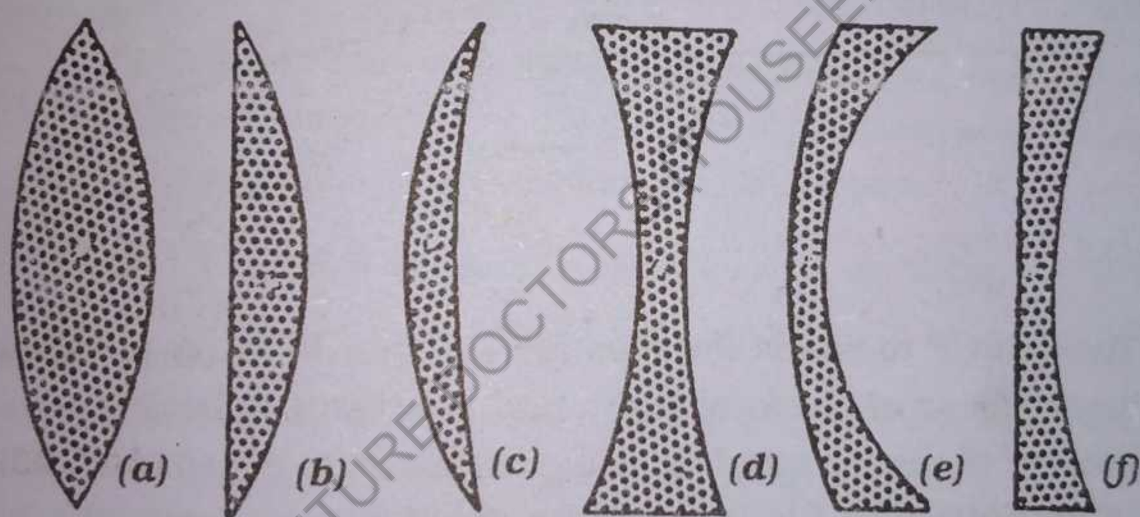


Fig: 10.1(a) Double convex lens (b) Plano- convex(c) concavo- convex

(d) Double concave (e) convexo- concave (f) Plano- concave

A convex lens is thicker in the middle and thiner at the edges, converges light rays towards its optical axis, (the line through its centre of curvature), so that a beam of parallel rays converges at a point F (Fig: 10.2). For example in bright sun light, a convex lens may produce a spot of light intense enough to ignite paper. A concave lens which is thiner in the middle and thicker at the edges bends rays outward from its optical axis Fig. 10.3.

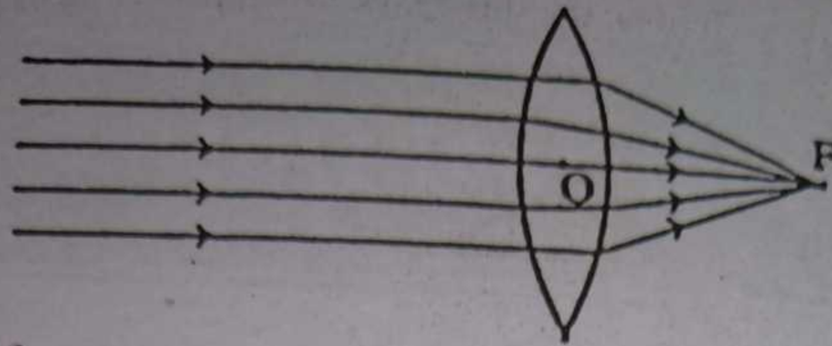


Fig: 10.2

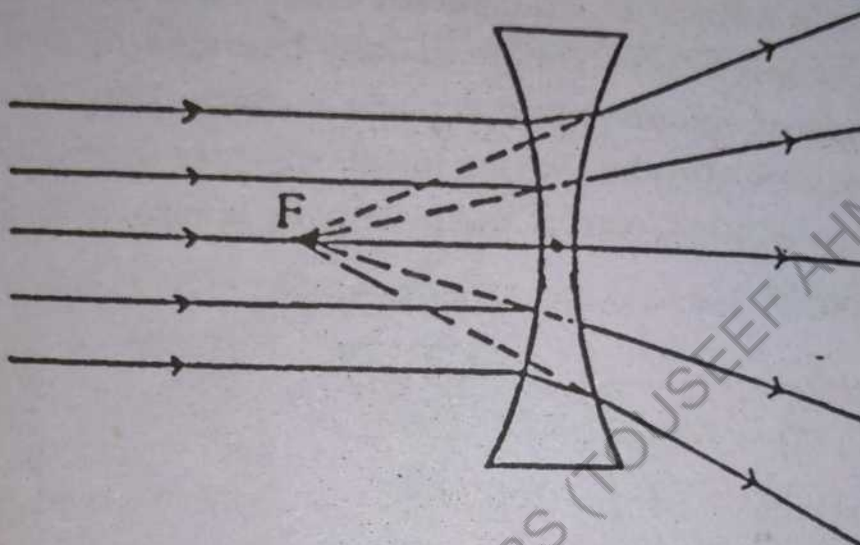


Fig: 10.3

The point F to which the rays are brought to focus is called the principal focus or the focal point, and the distance between the optical centre of the lens and its principal focus is called the focal length. Conventionally f is taken to be positive for convex lenses and negative for concave lenses.

10.2 IMAGE FORMATION

We have seen that in case of a convex lens light rays from a very distant point on its axis arrive parallel to the axis and meet to form an image at the principal focus. Rays from other points form images whose locations can be found graphically, if the focal length of the lens is known. However we cosider only three rays out of them whose intersection determine the location of the image as

shown in Fig. 10.4 . That is, the three rays which are numbered.

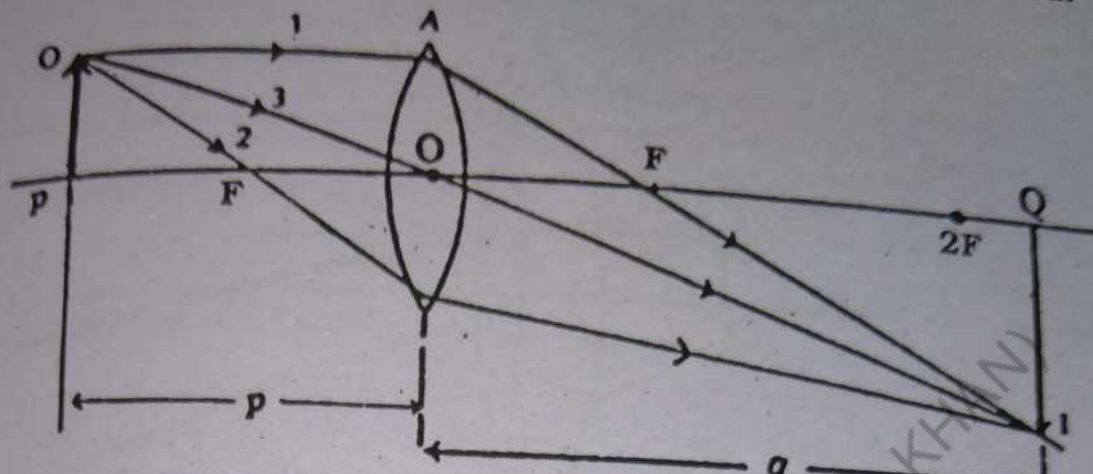


Fig: 10.4 Location of the rays in case of a convex lens

Fig. 10.4 (location of the rays in case of a convex lens) in ray diagram. Three rays are drawn from the tip of the object as follows:

(i) The ray 1 leaving the tip of the object parallel to the axis is refracted by the lens so that it passes through the principal focus F on the other side of the lens.

(ii) The ray 2 passing through the principal focus F emerges from the lens parallel to the axis.

(iii) The ray 3 passing through the optical centre of the lens remains unchanged in the direction.

The location of the image when an object is placed before a concave lens is shown in Fig. 10.5 . Convex lenses yield real image except for the case when the object is placed between the principal focus and the lens as shown in Fig. 10.6. Concave lenses yield

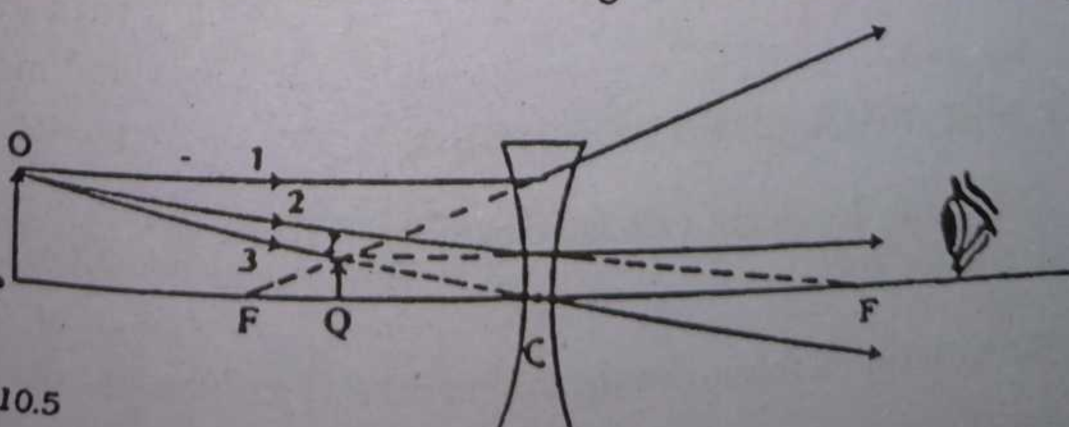


Fig: 10.5

only virtual images regardless of the position of the object. Real images are always inverted with respect to the object and virtual images are always errect.

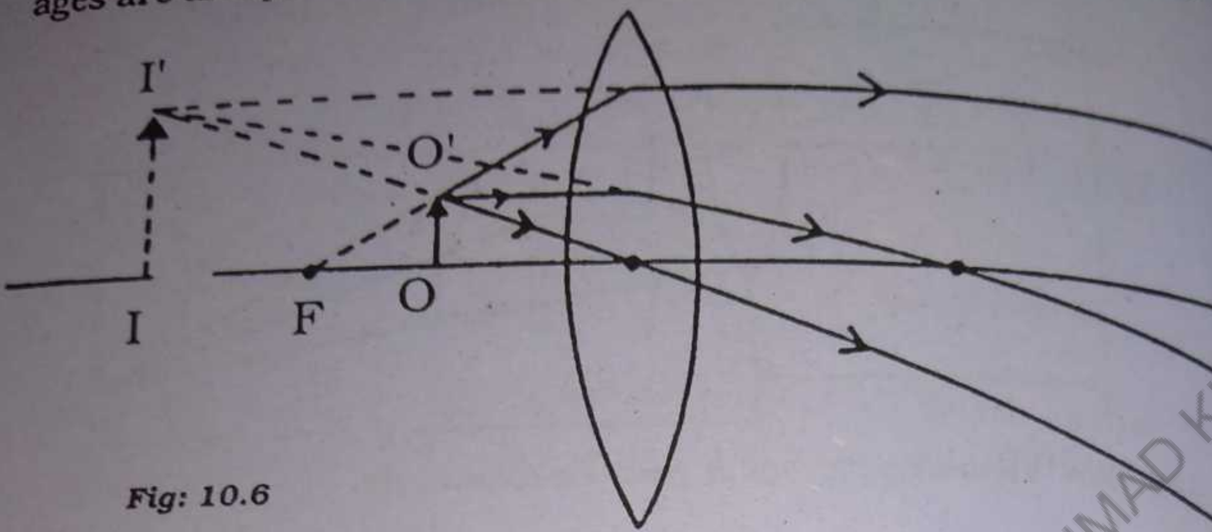


Fig: 10.6

The different forms of image formed, with respect to the object placed at the different positions are shown in Fig.10.7.

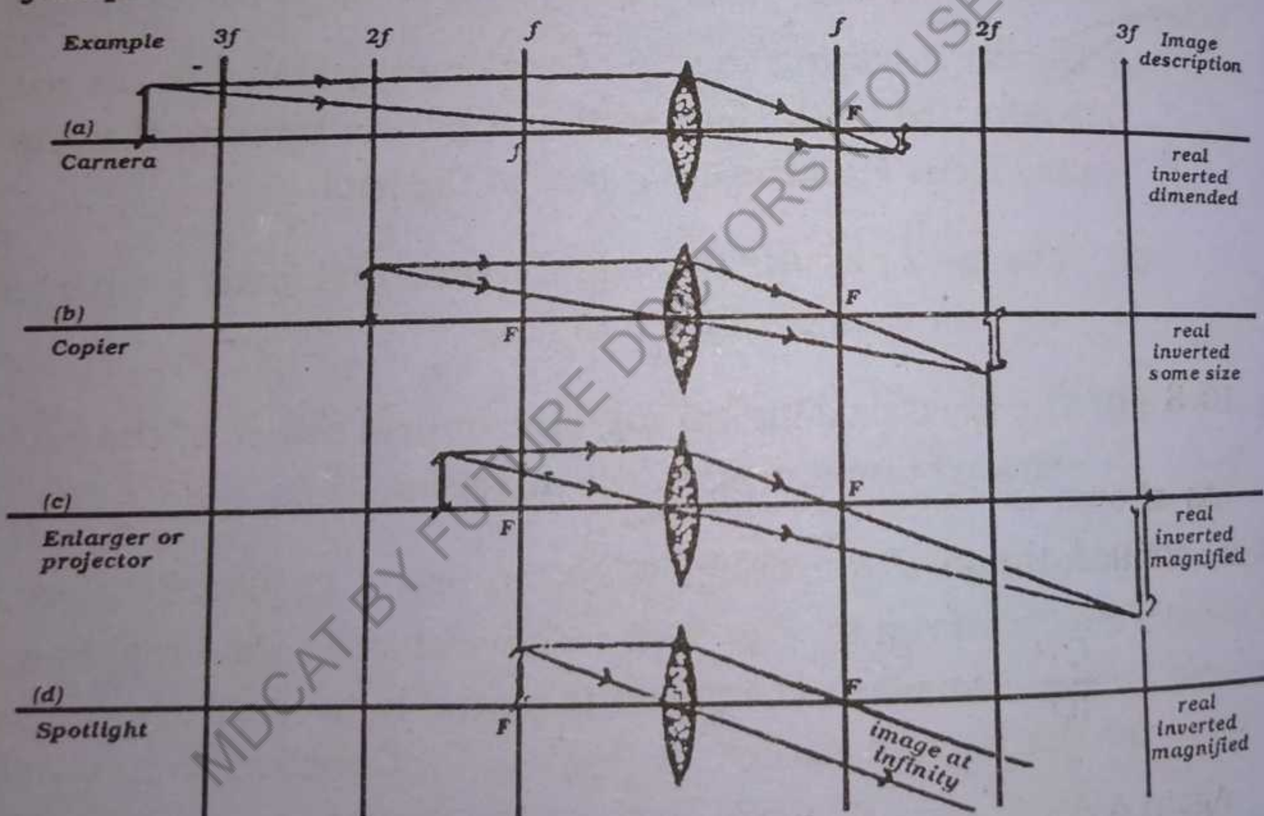


Fig: 10.7

10.3 THE THIN LENS FORMULA

Thin lens formula can be developed from the ray diagram shown in Fig. 10.8.

The distance from the optical centre of the lens to the object

is denoted by p. and the distance from the optical centre to the image is denoted by q. The following sign conventions are used.

Object distance p is $\begin{cases} + \text{ve for any real object.} \\ - \text{ve for any virtual object.} \end{cases}$

Image distance q is $\begin{cases} + \text{ve for a real image.} \\ - \text{ve for a virtual image} \end{cases}$

Focal length f is $\begin{cases} + \text{ve for a convex lens} \\ - \text{ve for a concave lens} \end{cases}$

Consider an object whose real and inverted image is formed by a thin convex lens as shwon in Fig 10.8.

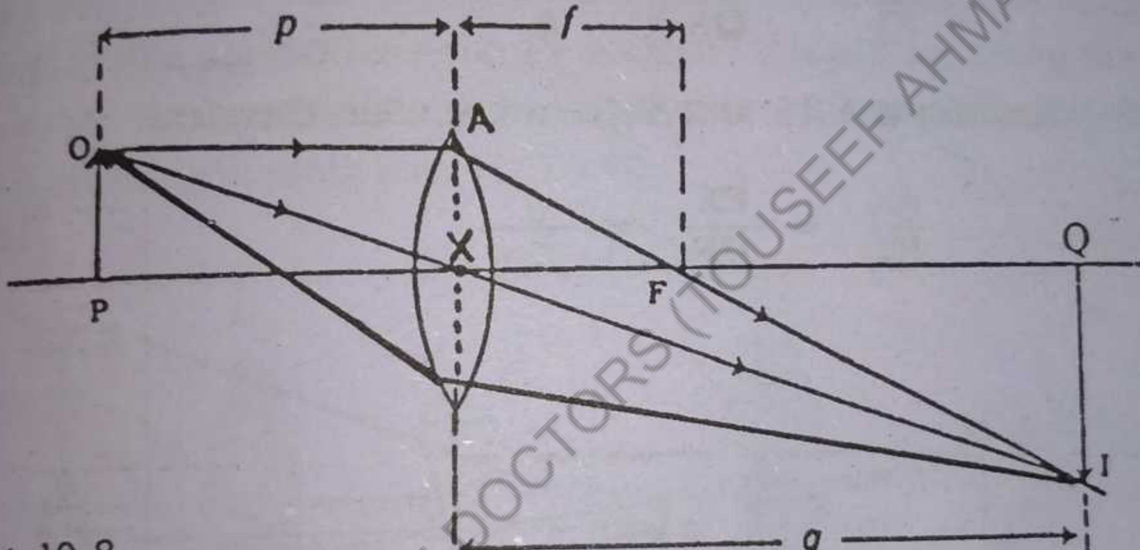


Fig. 10.8

As shown in the figure the right angled triangles OPX and IQX are similar, therefore we can write

$$\frac{OP}{IQ} = \frac{PX}{QX} = \frac{p}{q} \qquad 10.1$$

Again Δ AXF and Δ IQF are also similar.

$$\therefore \quad \frac{AX}{IQ} = \frac{XF}{QF} = \frac{f}{q-f} \qquad 10.2$$

Since AX = OP

$$\therefore \quad \frac{OP}{IQ} = \frac{XF}{QF}$$

or $$\frac{p}{q} = \frac{f}{q-f}$$ 10.3 (a)

simplifying the relation, we find that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$ 10.3, (b)

This is known as the lens equation or lens formula. In the same manner, we can obtain lens formula in case of a concave lens.

Consider the Fig. 10.9 , the triangles OPX and IQX are similar, therefore

$$\frac{OP}{IQ} = \frac{PX}{QX} = \frac{p}{q}$$ 10.4

Similarly Δ AXF and Δ IQX are similar, therefore

$$\frac{AX}{IQ} = \frac{FX}{QF} = \frac{f}{f-q}$$ 10.5

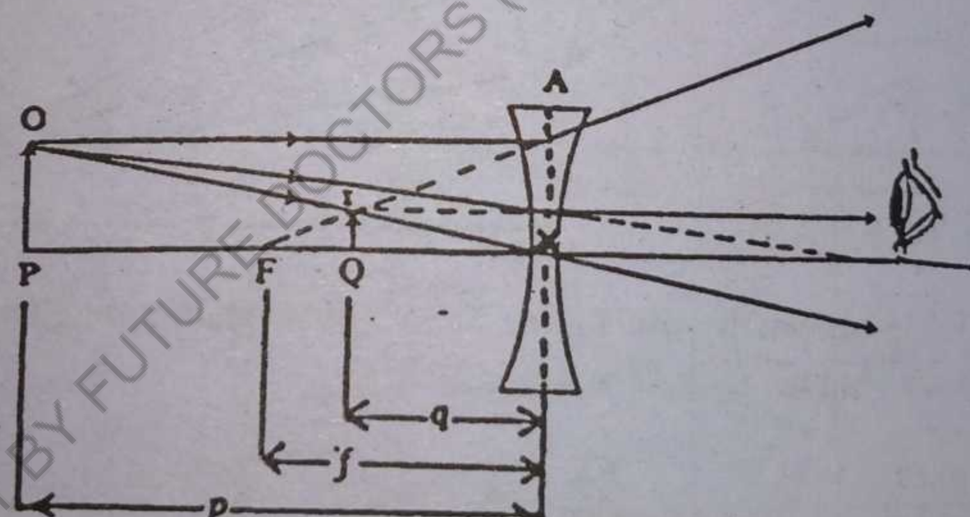


Fig. 10.9

Since AX = OP

$$\therefore \quad \frac{OP}{IQ} = \frac{FX}{QF}$$

$$\frac{p}{q} = \frac{f}{f-q}$$ 10.6

simplifying the above relation, we get

$$\frac{1}{p} - \frac{1}{q} = -\frac{1}{f}$$

Applying sign convention, the above equation becomes

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

10.4 COMBINATION OF THIN LENSES

In most of the optical instruments two or more lenses are used in combination. The location, size and nature of the final image can be determined by using the lens formula or ray diagram. In either case, we locate first the image formed by the first lens. Using that image as the object for the second lens, the final image formed by the second lens can be located. If there are more than two lenses, this process is continued; the object for each lens is the image for the preceding lens Fig 10.10

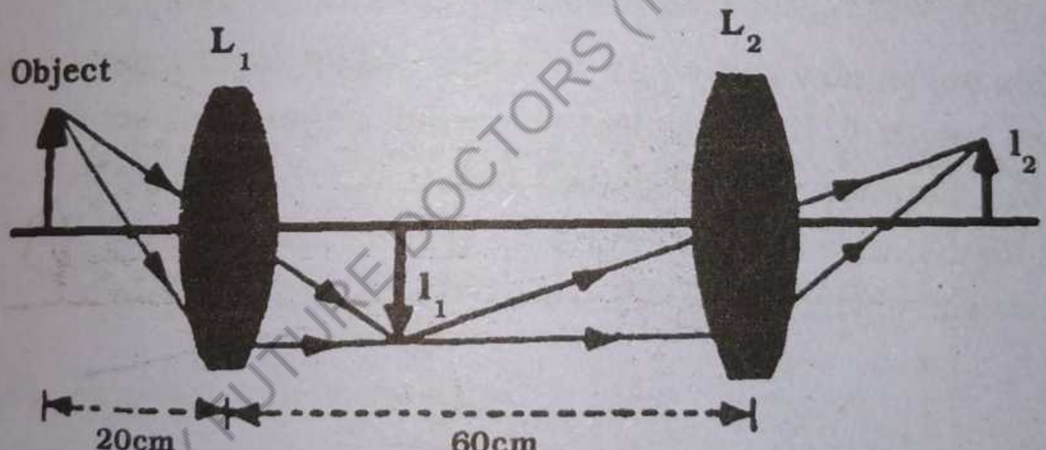


Fig. 10.10

Referring Fig. 10.10 we can see that lens L_1 forms an image I_1. This image acts as a real object for the Lens L_2, which forms a real image I_2. Notice that I_2 is inverted with respect to I_1 and erect with respect to the object.

We shall now consider the case when the two thin lenses are in contact, which means their separation is very small as compared to their focal lengths. This is illustrated in Fig. 10.11

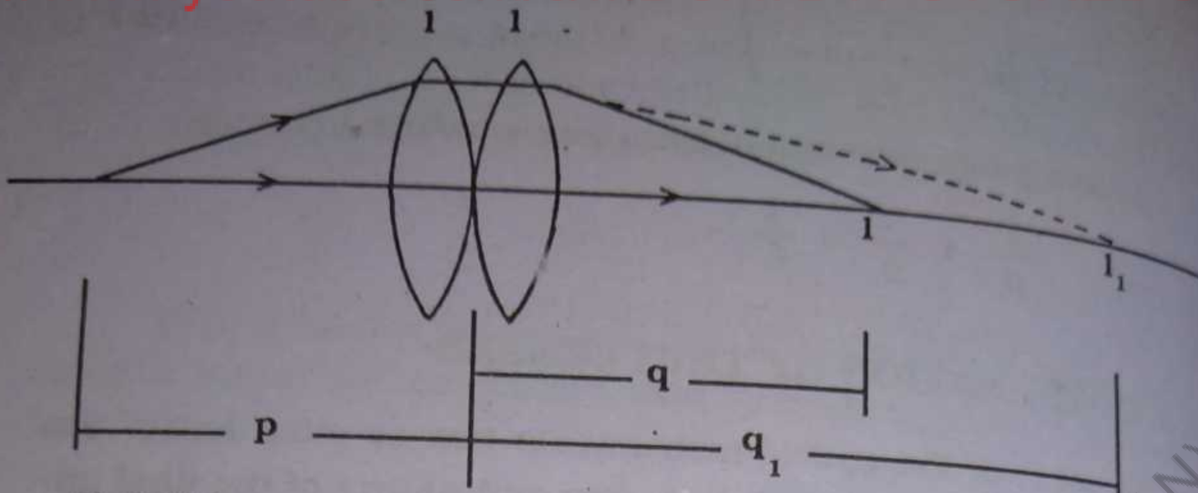


Fig:10.11

Let a point object O be placed at a distance p from the lens L_1 whose real image I_1 is formed by it at a distance q_1. From the lens formula

$$\frac{1}{p} + \frac{1}{q_1} = \frac{1}{f_1} \qquad 10.8$$

Where f_1 is the focal length of lens L_1.

This image now serves as a virtual object for the second lens L_2 of focal length f_2. If we neglect the small separation between the lenses, the distance of this virtual object from lens L_2 will be the same as its distance from the lens L_1. If the lens L_2 forms an image I of this virtual object at a distance q

$$-\frac{1}{q_1} + \frac{1}{q} = \frac{1}{f_2} \qquad 10.9$$

As the object is virtual for lens L_2, i.e $P_2 = -q_1$

Adding Eq. 10.8 and 10.9 , we get

$$\frac{1}{p} + \frac{1}{q_1} - \frac{1}{q_1} + \frac{1}{q} = \frac{1}{f_1} + \frac{1}{f_2} \qquad 10.10$$

$$\text{or } \frac{1}{p} + \frac{1}{q} = \frac{1}{f_1} + \frac{1}{f_2} \qquad 10.11$$

Now if we replace the two lenses of focal lengths f_1 and f_2 by a single lens of focal length f such that it forms an image at a dis-

tance q of an object placed at a distance p from it as shown in Fig.10.19, such a lens is called equivalent lens, and its focal length is known as equivalent focal length. For equivalent lens L, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \qquad 10.12$$

Comparing Eq (10.11) and (10.12), we get

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \qquad 10.13$$

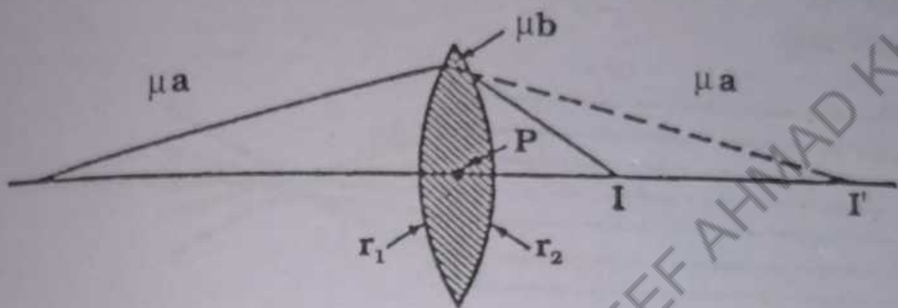


Fig: 10.12

This close combination behaves as a single lens whose focal length is given by the above relation.

The Eq. 10.13 shows that for a pair of lenses in contact the sum of the reciprocals of their individual focal lengths is equal to the reciprocal of the focal length of the combination.

10.5 POWER OF LENSES

If two lenses with focal lengths f_1 and f_2 placed in contact with each other are equivalent to a single lens with a focal length f satisfying $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ the power of the pair of lenses is $P = P_1 + P_2$. Thus the powers of lenses in contact are simply added to find the resultant power of the combination.

10.6 LENS ABERRATION

There are two main detects in the image formed by the lenses i.e. spherical abberation and chromatic aberration.

Spherical aberration is due to the fact that the focal points of light rays far from the optical axis of a spherical lens are different from those of rays passing through the centre. Fig. 10.13. illustrates the spherical aberration for parallel rays passing through a converging lens. Rays near the middle of the lens have longer focal lengths than rays at the edges. Hence there is no single focal length for a lens. Many cameras are equipped with an adjustable aperture to reduce spherical aberration when possible (An aperture is an opening that controls the amount of light transmitted through the lens).

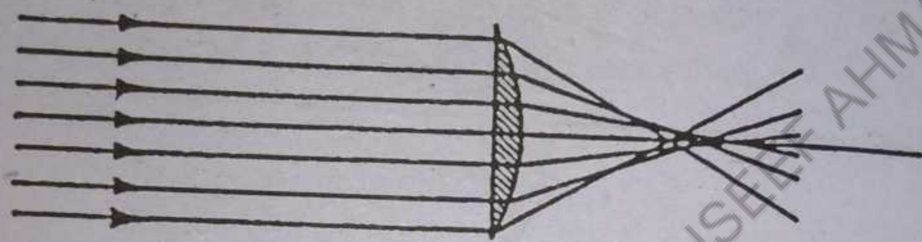


Fig: 10.13 (Spherical aberration)

The fact that different wavelengths of light refracted by lens focus at different points give rise to chromatic aberration.

From the lower diagram we see that the focal length of violet light is less than that of red light. Other wave lengths (not shown in figure) would have different focal points. This defect is called chromatic aberration. This aberration can be reduced to a greater

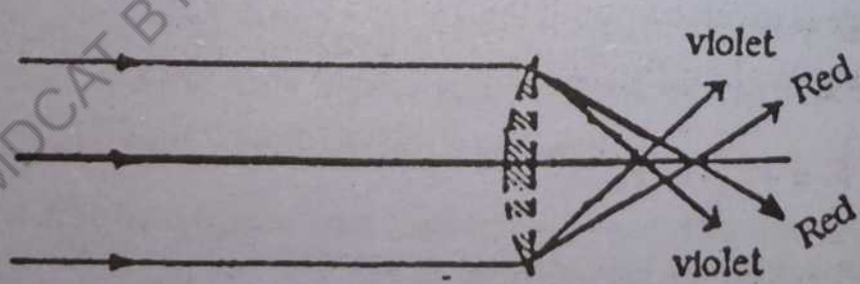


Fig 10.14 chromatic aberration

extent by the combination of convex and concave lenses made from the different type of glass.

10.7 LINEAR MAGNIFICATION

The Linear magnification, M, produced by a lens is defined as the ratio of the size of the image to the size of object. From Fig 10.15.

$$\text{Magnification} = \frac{\text{size of image}}{\text{size of object}}$$

$$M = \frac{IQ}{OP} = \frac{y'}{y}$$

A typical object and image linked by a ray through the pole is shown in fig 10.15. The ΔOPX and ΔIQX are similar and therfore,

$$\frac{IQ}{OP} = \frac{XQ}{XO} = \frac{q}{p}$$

$$\therefore \text{Magnification} = \frac{q}{p}$$

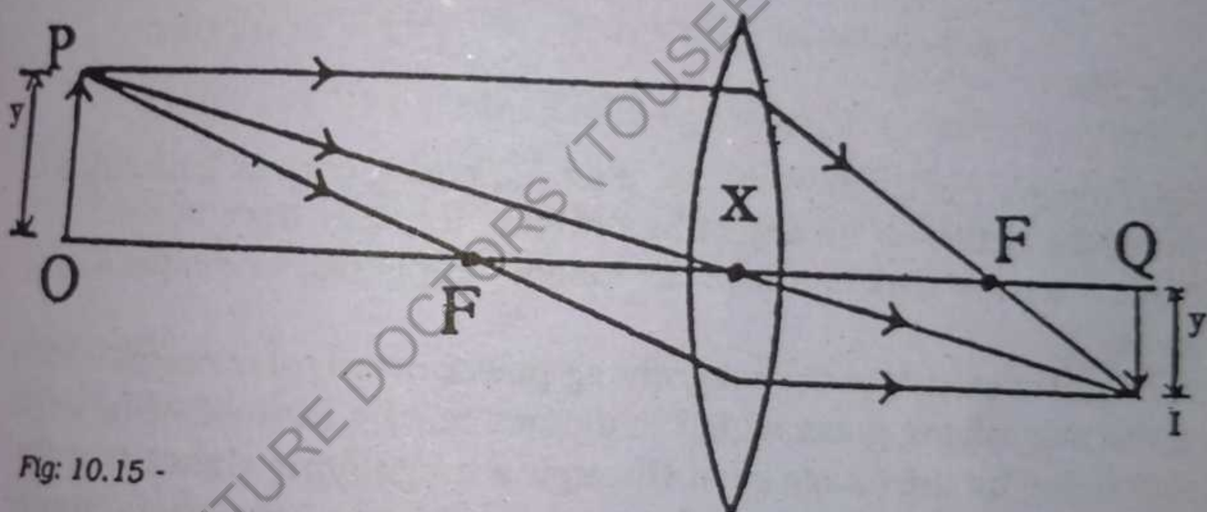


Fig: 10.15 -

If the image is inverted and if p and q are both positive, Y' and Y have opposite signs. then the formula for magnification becomes:

$$M = -\frac{q}{p}$$

In fact, this is just one of a series of sign convention associated with the lens equation and similar expressions.

10.8 MAGNIFYING GLASS

The apparent size of an object depends upon the angle subtended by it at the eye. This angle is called the visual angle.The

greater the visual angle, the greater is the apparent size of the object. This fact is illustrated in Fig 10.16 where the same object OP at two different positions will appear to be of different sizes at the eye. The smaller the distance of the object from the eye, the greater will be the visual angle consequently it will appear larger if we wish to see the fine details of a small object, we bring it as close to the eye as possible, thus increasing the visual angle and getting a large and real image on the retina of the eye. But we know that a normal person cannot see clearly an object if it is closer than the

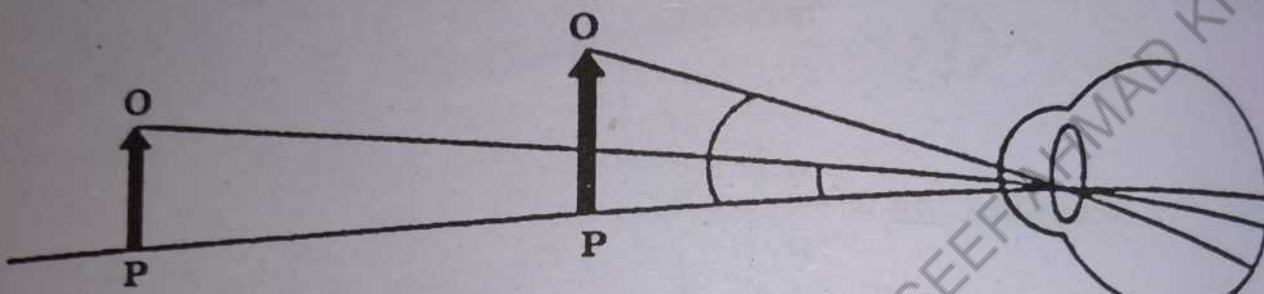


Fig: 10.16

least distance of distinct vision, d i.e. 25 cm. A convex lens helps us to see the details of an object by bringing it closer than 25cm. Such a convex lens is known as magnifying glass, or simply a magnifier.

Let us now calculate the magnifying power or angular magnification of the magnifying glass which is defined as the ratio of visual angle subtended by the image seen through a magnifying glass to the visual angle subtended by the object when placed at the least distance of distinct vision, when see through naked eye.

Consider a small object OP which is placed at a distance p within the focal length of a magnifying glass 'L' such that its erect; virtual and magnified image IQ is produced at the least distance of distinct vision 'd' as shown in Fig. 10.17(b). The magnifying power of the magnifying glass is given by

$$M = \frac{\beta}{\alpha} \qquad 10.14$$

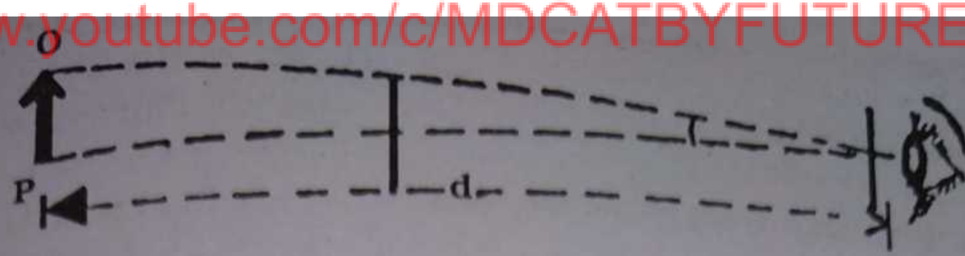
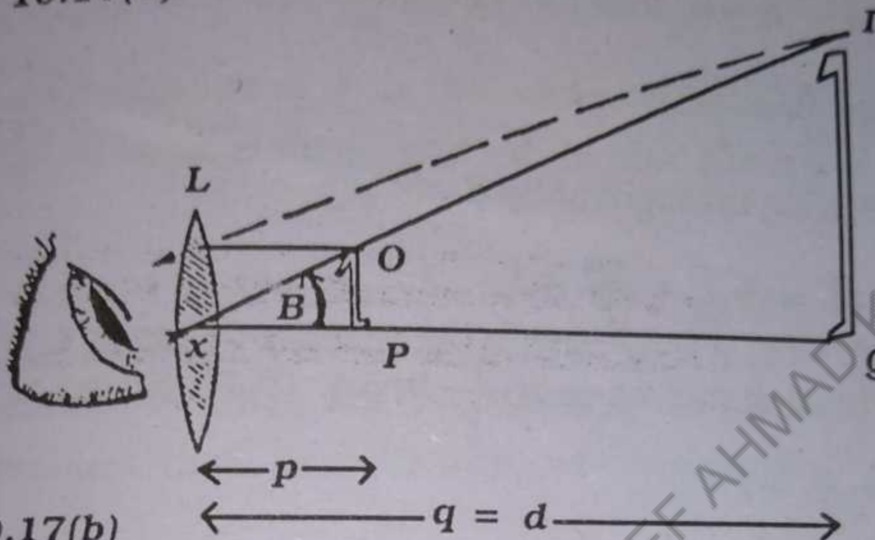


Fig: 10.17(a)



10.17(b)

Where α is the visual angle subtended by the object when placed at least distance of distinct vision, when seen through un-aided eye and β is the visual angle subtended by the image seen through the magnifying glass.

therefore

$$\tan \alpha = \frac{OP}{d} \quad \text{since } \alpha \text{ is small}$$

$$\therefore \tan \alpha = \alpha$$

$$\therefore \alpha = \frac{OP}{d} \qquad 10.15$$

In right angled ΔOPX in Fig. 10.17(b), we have

$$\beta = \tan \beta$$

$$\therefore \beta = \frac{IQ}{d} = \frac{OP}{P} \qquad 10.16$$

By substituting values of α and β from Eq. 10.15 and 10.16 in Eq.10.14, the angular magnification is

$$M = \frac{\beta}{\alpha} = \frac{d}{p} = \frac{IQ}{OP} = \frac{\text{Image Size}}{\text{Object Size}} \qquad 10.17$$

From the lens formula, we have

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

for the magnifying glass, we have

$p = +p$, $q = -d$ and $f = +f$

$$\therefore \frac{1}{p}-\frac{1}{d}=\frac{1}{f}$$

multiplying throughout by d, we get

$$\frac{d}{p}=\frac{d}{f}=+1=\frac{25}{f}+1$$

Substituting this value of $\frac{d}{p}$ in Eq. 10.17, we get

$$M=\frac{d}{f}+1 \qquad 10.18$$

Example 10.1

A chess piece 4cm high is located 10 cm, from the converging lens whose forcal length is 20 cm. What is the nature size and location of the image?

Solution:-

The object is between the focal point and the lens. The image distance is found by using the lens equation

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{q}$$

$$\frac{1}{20}=\frac{1}{10}+\frac{1}{q}$$

$$q=-20$$

Since q is negative, the image is virtual and therefore erect.

The magnification is given by

$$M = -\frac{q}{p} = \frac{20}{10} = +2$$

$$\therefore M = \frac{y'}{y} = -\frac{q}{p}$$

$$\text{or } y' = My = 2 \times 4 = 8 \text{ cm}$$

The magnification is 2 in this case, with final size being 8 cm. The positive value of image size means that the image is erect.

Example 10.2

How far from a convex lens whose focal length is + 20 cm must a specimen of a Red Admiral butterfly be placed if its image is to be a real one, three times as large as the object?

Solution:-

Although we neither know the distance of object nor that of the image, we can relate p and q from the magnification formula. Since the real image is inverted.

i.e $y' = 3y$, the magnification is given

$$\text{by } M = \frac{y'}{y} = \frac{-3y}{y} = -3$$

$$\text{Hence } M = \frac{-q}{p} \quad \text{or } q = +3p$$

The -ve sign is necessary since the image is to be real and therefore inverted. Now using the lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\text{or } \frac{1}{p} + \frac{1}{3p} = \frac{1}{20}$$

$$\therefore \frac{3+1}{3p} = \frac{1}{20}$$

or $3p = 4 \times 20 = 80$ cm

$p = 26.66$

Example 10.3

A convex lens of focal length 20 cm, is used to form an erect image which is twice as large as the object. Find the position of the object?

Solution:-

Magnification = + 2, as the image is erect

Thus $-\dfrac{q}{p} = +2$ or $q = -2p$

But $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$

Here $q = -2p$, $f = 20$ cm

Thus $\dfrac{1}{p} - \dfrac{1}{2p} = \dfrac{1}{20}$ cm

or $\dfrac{1}{2p} = \dfrac{1}{20}$

$p = 10$ cm

The object is 10 cm from the lens.

Example 10.4

An object is placed at a distance of 60 cm from a concave lens of focal length 30 cm. Find the position and nature of the image?

Solution:-

Here $p = 60$ cm, and $f = -30$ cm, as lens is concave

By the lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{60} + \frac{1}{q} = -\frac{1}{30}$$

form an erect position of the

$$\text{or } \frac{1}{q} = -\frac{1}{30} - \frac{1}{60} = -\frac{3}{60}$$

$$\therefore q = -\frac{60}{3} = -20cm$$

$$\text{Also magnification} = -\frac{q}{p} = -\frac{(-20)}{60} = +\frac{1}{3}$$

The negative sign of q shows that the image is virtual. The positive sign of magnification shows that the image is erect. Hence the image is erect, virtual and diminished.

Example 10.5

A diverging lens of focal length $f = -16$ cm is held 8 cm from a rare stamp. Where is the image of this stamp located and what is the magnification of the lens?

Solution:-

Given $f = -16cm$

$p = 8$ cm

$q = ?$

Using lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\therefore \frac{1}{q} = -\frac{1}{16} - \frac{1}{8} = -\frac{3}{16}$$

$$\text{or } q = -\frac{16}{3} = -5.33cm$$

from a concave ature of the im-

oncave

The $-ve$ sign shows that the image is virtual, hence erect and to the left of the lens. The magnification is

$$M = \frac{-q}{p} = -\frac{(-5.33)}{8} = +0.666$$

The positive value of magnification indicates that the image is erect. Since the magnification is less than 1, it means that the image is actually smaller than the object.

10.9 COMPOUND MICROSCOPE

A compound microscope is an optical instrument which is used to see small object with very high magnification. It consists of two convex lenses- an objective (a lens placed before the object) of very short focal length f_1, and an eye-piece (a lens placed before the eye) of relatively long focal length f_2. The objective forms a real, inverted and magnified image of the object just placed beyond its focus. The eye piece is used as a magnifying glass to see the image formed by the objective. The final image seen by the eye through the microscope is virtual and very much magnified. Fig. 10.18 shows the path of rays through the microscope.

In order to derive an expression for the magnifying power of microscope, consider a small object OP placed at the distance p just beyond the focus of the objective lens L_1, whose real, inverted and magnified image IQ is formed at a distance q from the objective lens.

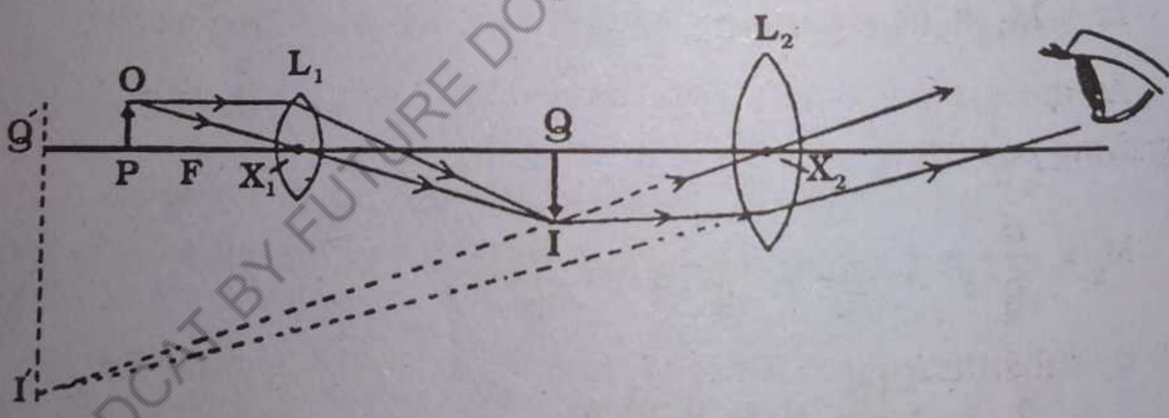


Fig 10.18 Compound Microscope

Magnifying power of the microscope is given by

$$M = \frac{\beta}{\alpha}$$

Where α and β are the visual angles subtended by the object and image respectively when formed at the least distance of distinct vision

$$\therefore \alpha = \frac{OP}{d} \quad \text{and} \quad \beta = \frac{I'Q'}{d}$$

Substituting the values of α and β in Eq: 10.19, we get

$$\therefore M = \frac{I'Q'}{OP}$$

$$\text{or } M = \frac{IQ}{OP} \times \frac{I'Q'}{IQ} \qquad 10.20$$

The magnifying power of objective L_1 is given by

$$M_1 = \frac{IQ}{OP} = \frac{q}{p} \quad (\because \Delta OPX_1 \text{ and } \Delta IQX_1 \text{ are similar})$$

The magnifying power of eye-piece is given by

$$M_2 = \frac{I'Q'}{IQ} \quad (\because \Delta IQX_2 \text{ and } \Delta I'Q'X_2 \text{ are similar})$$

The Eq. 10.20 can also be written as

$$M = M_1 \times M_2 \qquad 10.21$$

As the eye-piece acts here as a magnifying glass, hence its magnifying power can also be written as

$$M_2 = \frac{d}{f_2} + 1$$

By substituting values of M_1 and M_2 in Eq: 10.21 we get

$$M = \frac{q}{p}\left(\frac{d}{f_2} + 1\right) \qquad 10.22$$

As the object OP lies just outside the focus of the objective L_1

$$\therefore p \simeq f_1$$

Also the image IQ is formed very close to the eye piece L_2

$$\therefore QX_1 \simeq X_1X_2$$

or $q \simeq L$

Where L is the distance between objective and eye-piece, which is also called the length of the microscope. Hence the magnifying power of the compound microscope is found to be.

$$M = \frac{L}{f_1}\left(\frac{d}{f_2}+1\right) \qquad 10.23$$

Example 10.6

A microscope has an objective of 10.0mm focal length and eye piece of 25 mm focal length. What is the distance between the lenses and what is the magnification if the object is in sharp focus when it is 10.5mm from the objective?

Solution:

For the objective $p = 10.5\text{mm}$, $f_1 = 10\text{mm}$, $q = ?$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f_1}$$

$$\frac{1}{10.5} + \frac{1}{q} = \frac{1}{10}$$

$$\therefore q = 210 \text{ mm}$$

For the eye piece $f_2 = 25$ mm, $q = -250$ mm($\because$ the final image should be at the least distance of distinct vision)

$p = p_1 = ?$

$$\frac{1}{p_1} - \frac{1}{250} = \frac{1}{25}$$

$$p_1 = 22.7 \text{ mm}$$

Distance between the lenses

$$q + p_1 = 210 + 22.7 = 232.7\text{mm}$$

$$\text{Magnification by objective} = -\frac{q}{p} = \frac{210}{10.5} = -20$$

$$\text{Magnification by eye-piece} = \left(\frac{d}{f_2} + 1\right) = \left(\frac{250}{25} + 1\right) = 11$$

$$\text{Total magnification} = 20 \times 11 = 220$$

10.10 TELESCOPE

Telescopes are used to see distant objects.The image of a distant object formed by a telescope is smaller than the actual object; because it is much nearer to the eye and has a greater visual angle, the object looks larger when viewed through the telescope.

Astronomical Telescope

An astronomical telescope is needed to see the heavenly bodies i.e. planets and stars. It consists of two convex lenses an objective. L_1 of long focal length f_1 and an eye piece L_2 of short focal length f_2. Fig 10.19 shows the path of rays through an astronomical telescope.

In order to derive an expression for the magnifying power of the astronomical telescope consider a distant object whose real, inverted and diminished image IQ is formed by the objective L_1 at its focus.

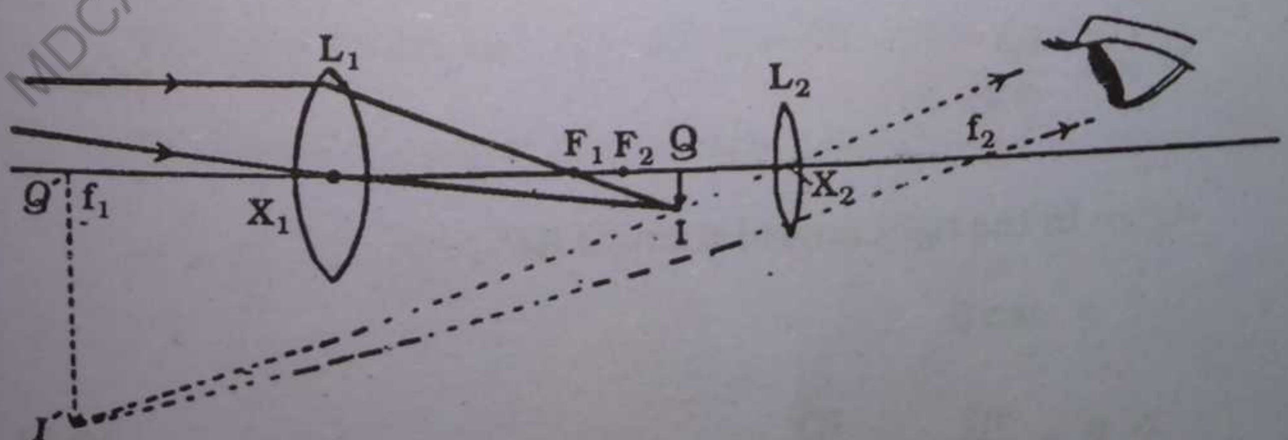


Fig. 10.19 Astronomical telescope.

The eye piece L_2 is so adjusted that IQ lies just inside its focus, hence a virtual and magnified image $I'Q'$ is formed by it at infinity.

$$\therefore QX_2 = f_2$$

Thus an astronomical telescope forms an inverted, virtual and highly magnified image of a distant object as shown in the Fig. 10.19 The distance between objective and eye piece is called the length of the telescope, which is given as

$$L = X_1 X_2$$

$$\text{or } L = QX_1 + QX_2$$

$$\text{Since } QX_1 = f_1 \text{ and } QX_2 = f_2$$

$$\therefore L = f_1 + f_2$$

The magnifying power of the telescope is given by

$$M = \frac{\beta}{\alpha} \qquad 10.25$$

Where α and β are the visual angles subtended by the object and image respectively.

In the right angled triangle IQX_1, we have

$$\alpha = \tan\alpha \ (\because \alpha \text{ is small})$$

$$\therefore \alpha = \frac{IQ}{QX_1} \qquad 10.26$$

Again in the right angled triangle IQX_2

$$\beta = \tan\beta$$

$$\therefore \beta = \frac{IQ}{QX_2} = \frac{IQ}{f_2} \qquad 10.27$$

Substituting the values of α and β from Eq.10.26 and 10.27 in Eq.10.25 we get

$$M = \frac{f_1}{f_2} = \frac{\text{focal length of objective}}{\text{focal length of eye piece}}$$

From the above relation, it is clear that for high magnification the focal length of the objective should be very large as compared to that of the eye piece

Example 10.7

An astronomical telescope has an objective lens whose power is 2 dioptres. This lens is placed 60 cm from the eye piece. When the telescope is adjusted for minimum eye strain. Calculate the angular magnification of the telescope.

Solution:-

Our first step is to determine the focal lengths of both lenses. For the objective we are given a power in dioptres

$$f_1 \text{ (in metres)} = \frac{1}{\text{power (in dioptres)}} = \frac{1}{2}\text{ m} = 50\text{cm}$$

The sum of the focal lengths $f_1 + f_2$ equals the separation of the two lenses, therefore

$$f_1 + f_2 = 60 \text{ cm and } f_2 = 60 - f_1 = 60 - 50 = 10 \text{ cm}$$

The magnifying power of this telescope is

$$M = \frac{f_1}{f_2} = \frac{50\text{cm}}{10\text{cm}} = 5$$

This is a low-power telescope that is used to examine a large area of the sky. We would require a high-power telescope to examine individual bodies, such as moon and planets in closed detail.

10.11 GALILEAN TELESCOPE

Galilean telescope is an optical instrument which is used to see the objects on earth. It was developed by Galileo. A Galilean Telescope consists of a convex lens L_1 as an objective and a concave lens L_2 as an eye piece. The virtual and erect image of distant object when seen through the eye piece L_2 is formed at the focus of the objective L_1.

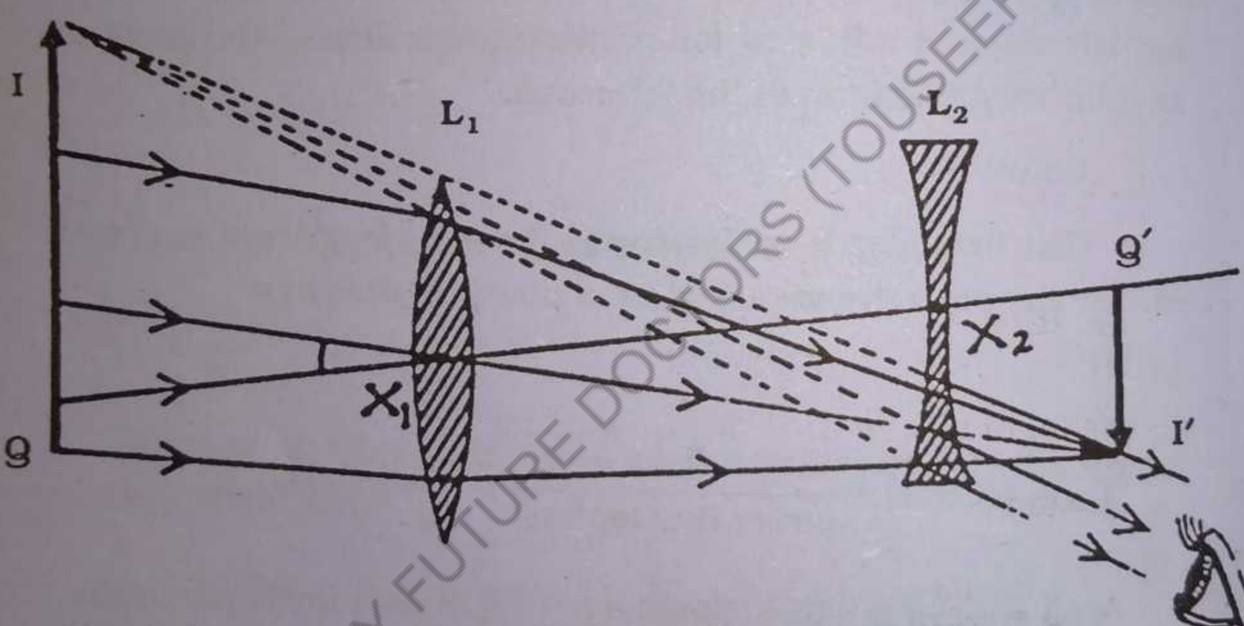


Fig 10.20

$$\therefore\ QX_1 = f_1 \qquad 10.28$$

The eye piece L_2 is introduced between L_1 and IQ such that IQ lies very close to its focus.

$$\therefore\ QX_2 = f_2 \qquad 10.29$$

Thus erect, virtual and highly magnified image I'Q' is produced as shown in the Fig.10.20 The distance between the objec-

tive and the eye piece is called length of the telescope, which is given by

$$L = X_1 X_2 = QX_1 - QX_2$$

$$\therefore L = f_1 - f_2$$

Expression for magnifying power

If α and β are visual angles subtended by the object and the image respectively, then magnifying power can be obtained as

$$M = \frac{\beta}{\alpha} \qquad 10.30$$

In the right angled ΔIQX_1, we have

$\alpha = \tan\alpha$ (∴ the angle is small)

$$\therefore \quad \alpha = \frac{IQ}{IX_1}$$

$$\text{or} \quad \alpha = \frac{IQ}{f_1} \qquad 10.31$$

Again in right angled ΔIQX_2, we have

$\beta = \tan\beta$

$$\therefore \quad \beta = \frac{IQ}{QX_2}$$

$$\text{or} \quad \beta = \frac{IQ}{f_2} \qquad 10.32$$

By substituting the values of α and β from Eq.10.39 and 10.40 in Eq. 10.38, we get

$$M = \frac{f_1}{f_2} = \frac{\text{focal length objective}}{\text{focal length of eye piece}}$$

10.12 TERRESTRIAL TELESCOPE

When telescope is used for astronomical purpose, i.e. when we see the heavenly bodies like moon and stars, their image is seen inverted. But when terrestrial objects are to be viewed, it is necessary to have an erect final image. The erection can be accomplished by introducing a third lens between the objective and eye piece of telescope. The arrangement is shown schematically in Fig. 10.21

The function of erecting lens is clear from Fig. 10.21. It only

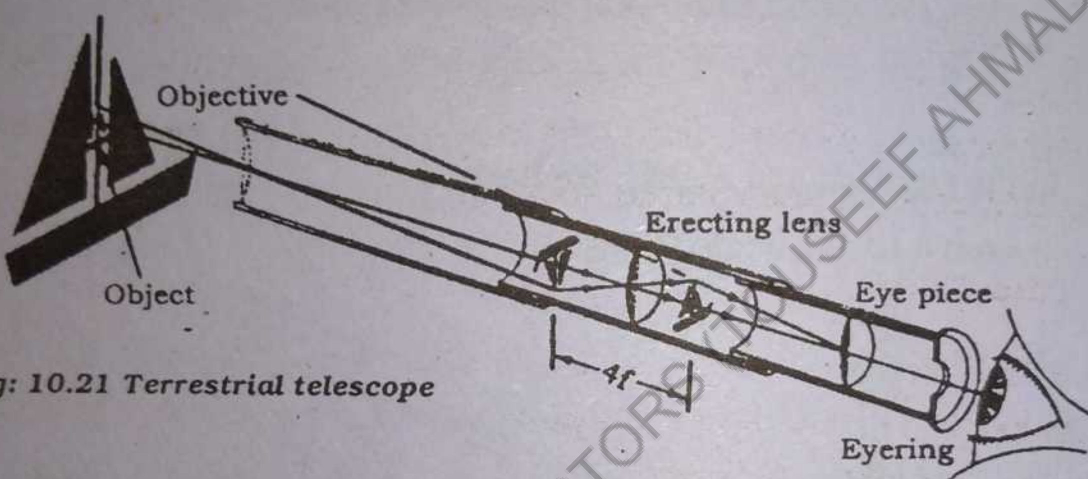


Fig: 10.21 Terrestrial telescope

inverts IQ to I″Q″ and does not change the magnifying power which is same as in the case of an astronomical telescope.

10.13 SPECTROMETER

It is an instrument which is used to study the spectrum of luminous bodies. The essential parts of this instrument are (a) collimator (b) telescope and (c) turn table as shown in the Fig 10.22

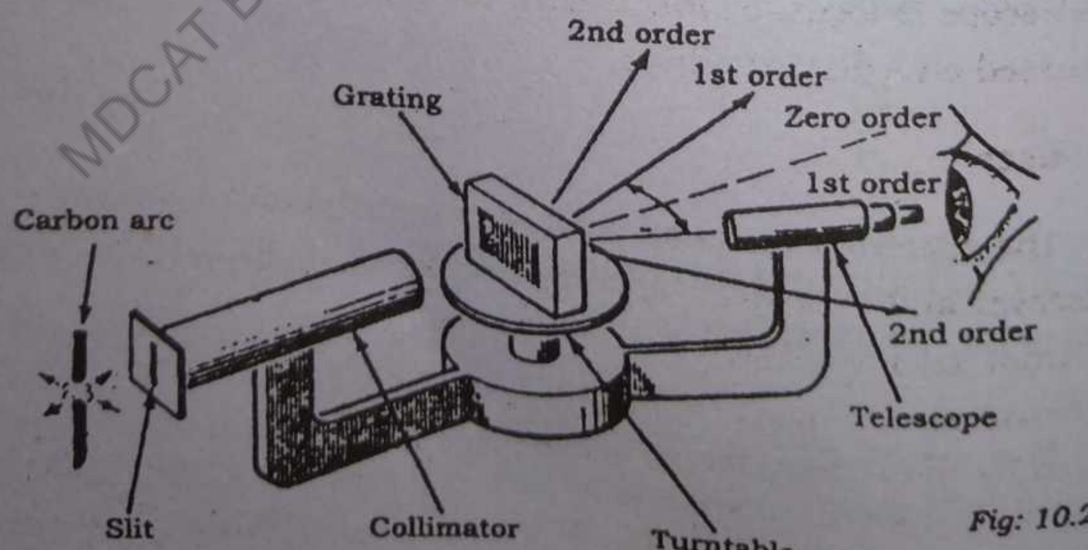


Fig: 10.22

(a) Collimator

It consists of a metal tube provided with a convex lens at its one end and an adjustable slit S at its other end. The length of the tube is altered by a screw to be equal to the focal length of the convex lens so that the collimator produces a parallel beam of light. The collimator is fixed to the base of the instrument while turn table and telescope can rotate about a common axis.

(b) Telescope

The telescope is simple astronomical telescope which is used for making measurements, or for examining the spectrum. The adjustment for focussing the telescope is made with the help of screw S_2. The telescope can be rotated about an axis.

(c) Turn Table

It is a circular metallic plate which can rotated about an axis. Its height is also adjustable and this can be levelled by means of three screws L,M and N.

There are arrangements for fine motion of the telescope and turn table. A vernier scale is provided to measure the angle with great accuracy, i.e in degrees and minutes.

Before using, the collimator is adjusted for parallel rays and the telescope is focussed for parallel rays or for infinity. For this it is focussed on a distant object.

Uses

The spectrometer is an analysing instrument used primarily to discover and measure the wave lengths of a given light. When light from an incandescent solid, liquid or gas is examined by a spectrometer, an image of the slit is formed for each wave length emitted by the source. Such a spectrum is called an emission spectrum. If all the visible wave lengths are present in the light which is analysed, the images overlap and form a continuous spectrum. If

the source emits only a few definite wave lengths, the images of the slit are separated from one another and appear as a series of bright lines. Such a spectrum is called a line spectrum. A third type of emission spectrum is called bandspectrum which shows band instead of lines. These bands are found to emit of closely spaced lines arranged in an orderly manner. The line spectrum of a mercury light is shown in Fig.10.23.

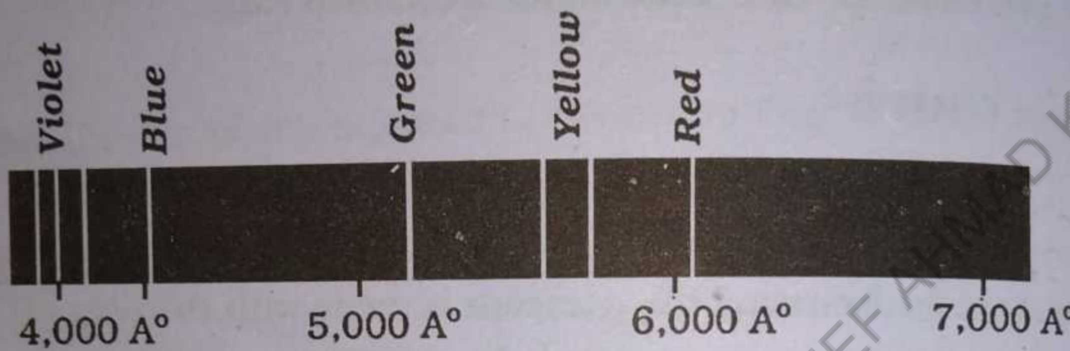


Fig: 10.23 Spectrum of mercury light

10.14 THE EYE

The eye is very important optical system which has much common with the camera. Like the camera the eye gathers light and produces a sharp image.

Fig.10.25 shows the essential parts of the eye. The front of the eye is covered by a transparent membrane called the cornea. This is followed by a clear liquid region, a variable aperture (iris and pupil) and the crystalline lens. Most of the refraction occurs in the cornea, since the liquid medium surrounding the lens has an average index of refraction close that of the lens. The iris is a muscular diaphragm that controls the size of the pupil. It regulates the

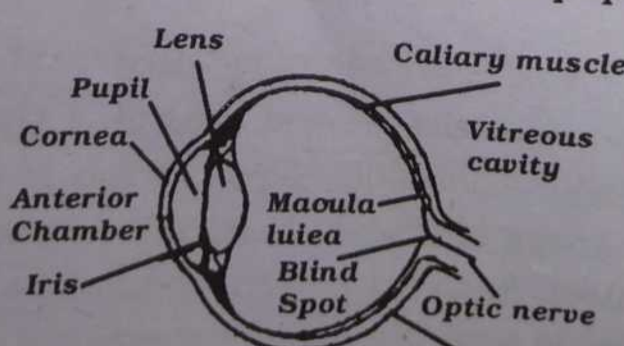


Fig: 10.24

amount of the light entering the eye by dilating the pupil in light of low intensity light. Light entering the eye is focused by the cornea-lens system on the back surface of the eye called the retina. The retina contains nerve fibres which branch out into millions of sensing structures called rods and cones. Optical image received by the retina is transmitted to brain via the optic nerve.

Although the eye is one of the most remarkable creation in nature, it often does not function perfectly. The eye may have several abnormalities, which can some times be corrected by eye glasses, contact lenses, or surgery. When the relaxed eye produces an image of a distant objected behind the retina, as in Fig 10.25 (a) the abnormality is known as myopia and the person is said to be farsighted. This condition is corrected with a converging lens, as shown in Fig. 10.25 (b). When an image of a distant object is focussed in front of the retina, as in Fig. 10.26 (a) the abnormality is known as hyperopia or short sightedness. This condition can be corrected with a diverging lens as in Fig 10.26(a), (b).

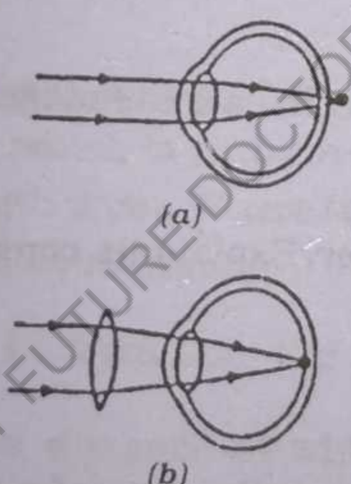


Fig: 10.25 (a) A hyperopic eye (far sighted) is slightly shorter than normal and so the image of a distant object focuses behind the retina. (b) The condition can be corrected with a converging lens.

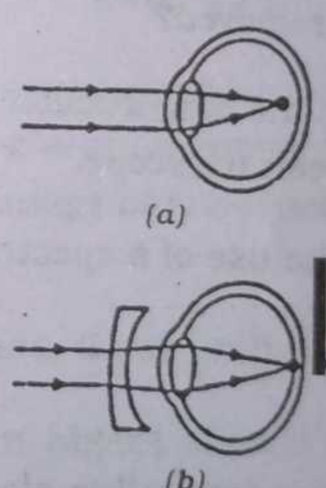


Fig: 10.26 (b). (a) A myopic eye (near sighted) is slightly longer than normal and so the image of a distant object forms in front of the retina. (b) The condition can be corrected with a diverging lens.

QUESTIONS

1. How can a real image be distinguished from a virtual image? Can each type of image be projected on screen? Explain.

2. Show how the position, nature and size of the image are formed by a (a) convex lens (b) concave lens.

3. Under what conditions does a converging lens act as a diverging lens?

4. Why is a convex lens of small focal length preferred for a magnifying glass?

5. Define the construction, working and magnifying power of a compound microscope.

6. How is the magnifying power of a (i) telescope and (ii) microscope affected by increasing the focal length of their objectives?

7. What is the difference between astronomical and terrestrial telescopes?

8. Explain the defects which occur in the lenses, and how they can be removed?

9. Explain the construction and calculate the magnifying power of Galilean telescope.

10. What is the use of a spectrometer? Explain its construction and working.

PROBLEMS

1. An object 4cm tall is placed near the axis of a thin converging lens. If the focal length of the lens is 25 cm, where will the image be formed and what will be the size of the image? Sketch the principal ray diagram.

(Ans. 100 cm behind 12 cm)

2. A convex lens has a focal length of 10 cm. Determine the image distances when an object is placed at the following distances from the lens.
50 cm, 20 cm, 15 cm, 10 cm, and 5 cm

(Ans. 12.5 cm, 20 cm, 30 cm, infinity and -10 cm)

3. Two converging lenses of focal lengths 40 cm and 50 cm are placed in contact. What is the focal length of this lens combination?What is the power of the combination in diopters?

(Ans.22.2 cm,4.5 diopters)

4. A converging lens of focal length 20 cm is placed infront of a converging lens of focal length 4 cm. What is the distance between the lenses if parallel rays entering the first lens leave the second lens as parallel rays?

(Ans. 24 cm)

5. A parallel light beam is diverged by a concave lens of focal length- 12.5 cm and then made parallel once more by a convex lens of focal length 50 cm. How far are the two lenses apart.

(Ans. 37.5 cm)

6. Two lenses are in contact,a converging one of focal length 30 cm and a diverging one of focal length -10 cm. What is the focal length and power of the combination?

(Ans. 15 cm, -6.7 diopters)

7. Moon light passes through a converging lens of focal length 19 cm, which is 20.5 cm from a second converging lens of focal length 2 cm. Where is the image of the moon produced by the lens combination?

(Ans. 14.5 cm from the 1st lens, 6 cm from the second lens)

8. Find the distance at which an object should be placed infront of a convex lens of focal length 10 cm to obtain an image of double its size?

(Ans. 15 cm for producing a real image, 5 cm for producing virtual image).

9. A compound microscope has a 12 mm focal length objective and a 75 mm focal length eye piece, and the two lenses are mounted 200 mm apart. If the final image is 225 mm from the eye piece, what is the magnification produced?

(Ans.44)

10. An astronomical telescope of angular magnifiction 1000 has an objective of 15 m focal length. What is the focal length of the eye piece?

(Ans. 15 mm)

11. A Galilean telescope has an objective of 120 mm focal length and an eye piece of 50 mm focal length. If the image seen by the eye is 300 mm from the eye piece, what is angular magnification?

(Ans: 2.66)

12. A compound microscope has an objective with a focal length of 10 mm and a tube 100 mm long. An image is produced 250 mm from the eye piece when the object is 12 mm from the objective. What is the angular magnification?

(Ans. 31)

13. A converging lens of 4 dioptres is combined with a diverging lens of -2 dioptres. Find the power and focal length of the combination.

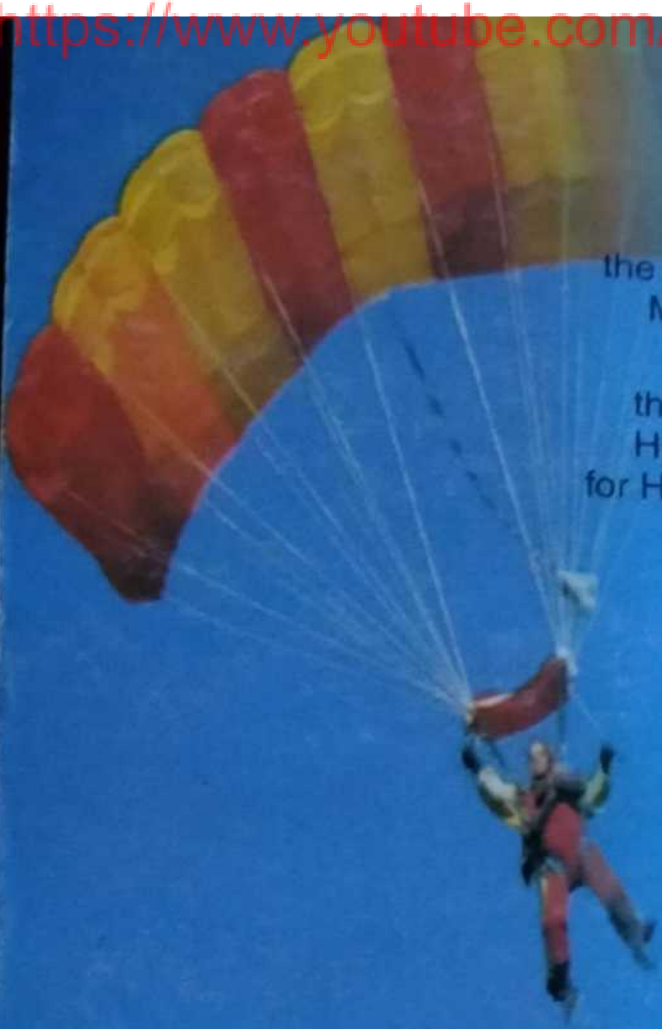
(Ans. Power of combination=2 dioptres; focal length of combination = 50 cm).

14. A convex lens forms image of an object on a fixed screen 20 cm from the lens. On moving the lens 60 cm towards the object, the image is again formed on the screen. What is the focal length of the lens?

(Ans: 16 cm)

15. Two converging lenses are 25 cm apart. Focal length of each is 10 cm. An object is placed in front of one lens at 50 cm. Find the distance between the object and the final image?

(Ans. 125 cm)



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